When do Local Operations and Classical Communication Suffice for Optimal Quantum State Discrimination?

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Abstract. In this paper we consider the conditions under which a given ensemble of two-qubit states can be optimally distinguished by local operations and classical communication (LOCC). For the well-known task of minimum error discrimination, it is shown that almost all two-qubit ensembles consisting of three pure states cannot be optimally discriminated using LOCC. This is quite surprising considering that *any* two pure states can be distinguished optimally by LOCC. Special attention is given to ensembles that lack entanglement, and we prove an easy sufficient condition for when a set of product states cannot be optimally distinguished by LOCC, thus providing new examples of the phenomenon known as "non-locality without entanglement". We then consider an example of N parties who each share the same state but who are ignorant of its identity. For any finite N, we prove that optimal identification of the state cannot be achieved by LOCC; however as $N \to \infty$, LOCC can indeed discriminate the states optimally. This is the first result of its kind. Finally, we turn to the task of unambiguous discrimination and derive new lower bounds on the LOCC inconclusive probability for symmetric states.

1 Introduction

The ability to distinguish one physical configuration from another lies at the heart of information theory. When quantum systems are used for information transmission, information is encoded into quantum states, and the processing of this information in a faithful manner requires the encoded states to be distinguishable from one another. Hence, a fundamental topic in quantum information is the problem of *state discrimination*, which investigates how well ensembles of quantum states can be distinguished under various physical conditions.

One important operational setting in which questions of distinguishability emerge is the so-called "distant lab" scenario. Here, some multiparty quantum state is distributed to spatially separated quantum labs, and the various parties use local measurements combined with classical communication to try and identify their state. This operational setting is also known as LOCC (Local Operations and Classical Communication), and the study of LOCC operations has played an important role in developing our understanding of not only quantum information processing, but also the nature of quantum entanglement itself.

As LOCC operations are just a subset of all possible physical operations, certain state discrimination tasks become impossible when the distant lab constraint is imposed. This limitation allows for the implementation of important information-theoretic objectives such as data hiding [7, 8] and secret sharing [9, 10]. However, in general it is a very challenging problem to decide whether or not a particular set of states can be optimally distinguished using LOCC. This is due to the complex structure of a general LOCC operation in which, due to the global communication, the choice of local measurement by one party at one particular round can depend on the measurement outcomes of all the other parties in previous rounds. It is thus helpful to visualize a general LOCC operation as a tree where each node indicates a particular choice of local measurement and each branch corresponds to a particular sequence of measurement outcomes. Deciding whether or not a certain discrimination task is feasible by LOCC therefore amounts to a consideration of all such possible trees.

Despite its complexity, partial progress has been made in understanding the capabilities and limitations of LOCC state discrimination. Most notably is the discovery that *any* two orthogonal pure states can be perfectly distinguished using LOCC [11]. A similar result holds for pairs of non-orthogonal states in which again, LOCC can obtain the optimal discrimination success probability that is physically possible [15]. This finding is particularly relevant to the current paper since we will show that, in sharp contrast, almost all triples of two qubit states *cannot* be optimally distinguished by LOCC.

The fact that non-LOCC measurements can distinguish certain ensembles better than any LOCC strategy may not be overly surprising when the ensemble

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states possess entanglement. This is because entanglement embodies some non-local property of two or more system, and thus some global measurement across all systems is needed in general to discriminate among entangled states. However, rather surprisingly, certain ensembles exist that consist of unentangled states that cannot be distinguished optimally using LOCC [13]. This phenomenon is often called "nonlocality without entanglement," and it essentially reflects that fact that nonlocality and entanglement are two different physical properties of multipartite quantum systems. Understanding the difference between the two is an important problem in quantum information science, and thus a main objective of this paper is to study, in particular, LOCC discrimination of product state ensembles.

2 Summary of our results

In this paper, we begin by returning to the problem of *perfect* state discrimination among two-qubit orthogonal states. While our primary interest is LOCC discrimination, we will also consider discrimination by the more general class of separable operations (SEP). The two-qubit perfect discrimination problem has been solved for almost all types of ensembles[1]. Our first contribution is that we solve the missing piece of perfect discrimination between one pure state and one mixed state by either LOCC and SEP (Theorem 1 below). Interestingly, we find that SEP is more powerful than LOCC in the sense certain ensembles are distinguishable by SEP but not LOCC. This result is important since it allows us to construct in examples of one pure product state and one separable mixed state that cannot be optimally distinguished by LOCC. Thus, we obtain a class of two state ensembles which demonstrate nonlocality without entanglement.

We next move on to investigate the problem of *minimum-error* discrimination between linearly independent states. However, we prove that this seemingly more general problem actually reduces to the problem of perfect discrimination of orthogonal states (Proposition 2). This reduction therefore allows us to apply the results of perfect discrimination among orthogonal toward the problem of minimum-error discrimination of non-orthogonal (linearly independent) states. Consequently, we obtain our main result that almost any three states cannot be optimally distinguished by LOCC. More precisely, if we select a three state ensemble by randomly choosing our states, then almost surely will LOCC fail to discriminate them as well as a more general global measurement. We also considers two-qubit product state ensembles. We are able to obtain a simple necessary condition for when three product states cannot be distinguished optimally by LOCC (Theorem 5). With this result, new examples of nonlocality without entanglement can easily be constructed.

We also move beyond two qubit ensembles and consider the optimal discrimination of three symmetric N-qubit states. The specific ensemble we analyze is the N-copy generalization of the celebrated double trine ensemble [19]. We prove that for any finite N, the ensemble cannot be optimally discriminated using N-party LOCC. However as $N \to \infty$, we give a protocol that indeed achieves optimal (perfect) discrimination (For proof, see Sec. IV.B.2 of the attached supplemental material). This is quite different from the N-copy discrimination among two possible states which can always be accomplished optimally by LOCC [21].

Finally, we consider the task of unambiguous discrimination by LOCC. We obtain new upper bounds on the success probability obtainable by any LOCC measurement for a set of linearly independent symmetric states as a function of the *a priori* state probabilities (Theorem 6). With this simple examples can be found when LOCC is insufficient for optimal unambiguous discrimination.

[1]See Sec 3 of the attached full paper for detailed discussion.

3 Technical Propositions and Theorems

Our first result regards the condition for perfect distinguishability by separable and LOCC operations between two bipartite quantum states, one pure and the other mixed.

Theorem 1 *Let $\{|\Psi\rangle, \rho\}$ be two orthogonal states ($tr[|\Psi\rangle\langle\Psi|\rho] = 0$) on $\{|\Phi\rangle\}^{\perp}$. Then $|\Psi\rangle$ and ρ are perfectly distinguishable if and only if i) the matrix $\Psi\Phi^{-1}$ has two antiparallel eigenvalues; and ii) $C(\Psi) \leq C(\Phi)$. In particular, when Φ is a maximally entangled state, any such $|\Psi\rangle$ and ρ are perfectly distinguishable!*

For LOCC, the states are perfectly distinguishable if and only if either $|\psi\rangle$ is a product state, or condition (i) is satisfied and equality holds for condition (ii).

In the following, we have a series of results for the optimal discrimination of $2 \otimes 2$ linearly independent states under the measure *minimum error probability.* These results are obtained based on the following powerful proposition.

Proposition 2 *Let $\mathcal{E} = \{\rho_i, p_i\}_{i=1}^{n}$ be an ensemble of linear independent states; i.e. for spectral decompositions $\rho_i = \sum_{j=1}^{r_i} \lambda_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|\psi_{ij}$, the $|\psi_{ij}\rangle$ are linearly independent. Let S be the subspace spanned by the $|\psi_{ij}\rangle$, and let P_{opt} be the optimal minimum error probability in discrimination. Then there exists a unique decomposition of $S = S_1 \oplus S_2 \oplus ... \oplus S_n$ with S_i having dimension r_i such that a POVM can obtain P_{opt} on $\mathcal{E}$ if and only if it can* *perfectly distinguish the normalized subspace projectors* $\frac{1}{r_1}\Upsilon_{S_1}$, $\frac{1}{r_2}\Upsilon_{S_2}$,..., $\frac{1}{r_2}\Upsilon_{S_n}$.

In the pure state case, the subspace projectors Υ_{S_i} correspond to an orthonormal basis $\{|\phi_i\rangle\}_{i=1}^n$ for the space spanned by the $|\psi_i\rangle$. Thus, any LOCC POVM Π_i optimally distinguishes the $|\psi_i\rangle$ if and only if it can perfectly distinguish the $|\phi_i\rangle$. However for two-qubit ensembles, the conditions for perfect discrimination among orthogonal states have already been proven by Walgate and Hardy [11]. We thus obtain the following.

Proposition 3 *Consider an ensemble of linearly independent two-qubit states* $\{|\psi_i\rangle, p_i\}_{i=1}^n$*. If* $n = 3$*, then an LOCC protocol can optimally discriminate the ensemble (in the minimum error sense) if and only if the states* $\{|\phi_i\rangle\}_{i=1}^n$ *corresponding to the projectors* $\Upsilon_{S_i} = |\phi_i\rangle\langle\phi_i|$ *described by Proposition 2 contain at least two product states. If* $n = 4$*, then all of the* $|\phi_i\rangle$ *must be product states.*

Applying this result is still rather difficult since there appears to be no easy method for determining whether or not the optimal POVM projectors $|\phi_i\rangle\langle\phi_i|$ have product state form. However, by envoking a probabilistic argument, we can prove a very strong result, which is one of the main contributions of our work. It is quite surprising considering that any two pure states, *orthogonal or not*, can be distinguished optimally by LOCC.

Theorem 4 *Three randomly chosen two-qubit pure states almost surely cannot be optimally discriminated by LOCC.*

When we restrict our attention to distinguishing only product pure states, we can obtain a necessary condition for optimal discrimination by LOCC.

Theorem 5 *Suppose that* $\{|\psi_\lambda\rangle := |\alpha_\lambda\rangle|\beta_\lambda\rangle, p_\lambda\}_{\lambda=1}^3$ *(*$p_\lambda > 0$*) is some linear independent two-qubit product state ensemble that spans* $\{|\Phi\rangle\}^\perp$*. Let* $\lambda_{min}(\Phi)$ *denote the smallest squared Schmidt coefficient of* $|\Phi\rangle$*. If*

$$p_i^2\lambda_{min}(\Phi) > p_j^2|\langle\psi_i|\psi_j\rangle|^2 + p_k^2|\langle\psi_i|\psi_k\rangle|^2$$

for every choice of i, j, k *such that* $\{i, j, k\} = \{1, 2, 3\}$*, then the ensemble cannot be distinguished optimally (in the minimum error sense) with LOCC.*

Theorem 5 is very useful for constructing ensembles that demonstrate "non-locality without entanglement". Despite consisting of product states, ensembles satisfying the condition of Theorem 5 possess some non-local aspect since LOCC is insufficient for optimal discrimination. Furthermore, we can obtain examples in which separable operations attain optimal discrimination but LOCC cannot. For this, we rely on a result from [16] that three states can be perfectly distinguished by separable operations iff their concurrence sums to the concurrence of their common orthogonal complement state. Hence, separable opreations becomes strictly more powerful than LOCC for distinguishing a set of product states $|\psi_i\rangle$ that satisfy Theorem 5 when their corresponding detection states $|\phi_i\rangle$ satisfy $\sum_{i=1}^3 C(\phi_i) = C(\Phi)$.

An important example of such an ensemble is the so-called "double trine" ensemble [19], which is given by a uniform distribution of the states $|\psi_i\rangle = |s_i\rangle \otimes |s_i\rangle$ for $i = 0, 1, 2$ where

$$\begin{aligned} |s_0\rangle &= |0\rangle \\ |s_1\rangle &= -1/2|0\rangle - \sqrt{3}/2|1\rangle \\ |s_2\rangle &= -1/2|0\rangle + \sqrt{3}/2|1\rangle. \end{aligned} \quad (1)$$

The inability for LOCC to optimally discriminte the double trine states follows from Theorem 5 and the fact that $|\langle\psi_i|\psi_j\rangle|^2 = 1/16$, while $\lambda_{min}(\Phi) = 1/2$. Thus, $1/2 > 1/8$. On the other hand, it can be easily computed that the detection states $|\phi_i\rangle$ each have a concurrence of 1/3. Since the maximally entangled singlet state lies orthogonal to each of the $|\psi_i\rangle$, we have indeed have $\sum_{i=1}^3 C(\phi_i) = C(\Phi) = 1$. Hence, a separable POVM can optimally discriminate the double trine ensemble.

Our next major result involves generalizing the double trine discrimination problem to N parties. Specifically, we suppose that N copies of $|s_i\rangle$ are distributed to N different parties and their goal is to identify which state they possess by N-party LOCC. We first show that this problem is essentially identical to the "lifted trine" problem studied by Peter Shor [20]. We then compute the optimal *global* measurement for this task and prove that for any finite N, an LOCC measurement is unable to attain the optimal global success probability. This result is quite interesting when one considers the N-copy problem for two pure state ensembles. It has been proven that N-party LOCC can always obtain the optimal success probability [21]. However, we further show that asmptotically (i.e. $N \to \infty$), the N-copy trine states *can* be discriminated optimally by LOCC.

Our last result regards the conditions for optimal unambiguously discriminating linearly independent states. We obtain an upper bound on the LOCC conclusive probability for three symmetric states. By symmetric states, we mean those that are invariant under the SWAP operation $\mathbb{F}$, which acts on any product state $|\alpha\beta\rangle$ by $\mathbb{F}|\alpha\beta\rangle = |\beta\alpha\rangle$.

Theorem 6 *Let* $\{|\psi_i\rangle, p_i\}_{i=1...3}$ *be an ensemble of linearly independent symmetric pure states with dual basis* $|\widetilde{\psi_i}\rangle$*. If* $C(\widetilde{\psi_i}) \geq |\langle\widetilde{\psi_i}|\psi_i\rangle|^2$ *for all* i*, then LOCC cannot obtain an unambiguous probability greater than* p_{max}*.*

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