Construction and Physical Realization of POVMs on a symmetric subspace

[1,3]S. P. Shilpashree, [2] Veena Adiga, and [3]Swarnamala Sirsi

[1]*K.S. School of Engineering and Management, Bangalore*

[2]*St. Joseph's College(Autonomous), Bangalore and*

[3]*Yuvaraja's College, University of Mysore, Mysore*

(Dated: June 16, 2013)

I. INTRODUCTION

Positive-operator-valued-measures, POVM's are the most general class of quantum measurements, of which von Neumann measurements are merely a special case. The rapidly developing quantum information theory has generated a lot of interest in the construction and experimental implementation of POVMs [1–3]. Some tasks that cannot be performed using projective measurements can be completed using POVMs. For example, a set of non-orthogonal states cannot be distinguished using projective measurements, but can be discriminated unambiguously using POVMs [4, 5]. This concept can be used to construct POVMs in conclusive teleportation [6, 7]. Recently, POVM instead of usual projective measurement has been utilized to realize some quantum communication protocols [8–12]. In the context of RSP, POVM has already attracted a lot of attention [13]. Ram A. Somaraju, Alain Sarlette and Hugo Thienpont [14] have shown how quantum filtering can be performed in the POVM setting.

In this paper we introduce the most general method of constructing POVMs in terms of detection operators, on N+1 dimensional symmetric space which is a subspace of 2N dimensional Hilbert space in which N qubits reside. A set of N-qubit pure states that respect permutational symmetry are called symmetric states. Symmetric space can be considered to be spanned by the eigen states $|jm>$, $-j \leq m \leq +j$ of angular momentum operators J^2 and J_z, where $j = N/2$. A large number of experimentally relevant states [15] possesses symmetry under particle exchange, which significantly reduces the computational complexity. The detection operators are found to be the well known irreducible tensor operators τ^q_k which are used for standard representation [16] of spin density matrices. Further, by invoking Neumark's theorem [17], we demonstrate the physical implementation of our symmetric two-qubit POVMs as projection operators in the two qubit tensor product space.

II. OPERATOR REPRESENTATION IN TERMS OF IRREDUCIBLE TENSORS

The systematic use of tensor operators was first suggested by Fano[16]. In this representation, any Hermitian matrix 'H' in the symmetric space of dimension $N + 1$ in terms spherical tensor parameters [16] can be expressed as

$$H(\vec{J}) = \sum_{k=0}^{2j} \sum_{q=-k}^{+k} h_q^k \, \tau_q^{k^\dagger}(\vec{J}) \tag{1}$$

where $\tau_q^k{}'s$ (with $\tau_0^0 = I$, the identity operator) are the irreducible (spherical) tensor operators of rank 'k' in the (2j+1) dimensional spin space. The operators $\tau_q^k{}'s$ are homogeneous polynomials of rank 'k' and the projection 'q' is constructed out of angular momentum operators $J_x,\ J_y,\ J_z$. In particular, following the well known Weyl construction [[18, 19]] for $\tau_q^k{}'s$ in terms of angular momentum operators J_x, J_y and J_z, we have

$$\tau_q^k(\vec{J}) = \mathcal{N}_{kj}\, (\vec{J}\cdot\vec{\bigtriangledown})^k\, r^k\, Y_q^k(\hat{r})\,, \tag{2}$$

where

$$\mathcal{N}_{kj} = \frac{2^k}{k!}\sqrt{\frac{4\pi(2j-k)!(2j+1)}{(2j+k+1)!}}, \tag{3}$$

are the normalization factors and $Y_q^k(\hat{r})$ are the spherical harmonics.

The $\tau_q^k{}'s$ satisfy the orthogonality relations

$$Tr(\tau_q^{k^\dagger}\tau_{q'}^{k'}) = (2j+1)\,\delta_{kk'}\delta_{qq'}\,. \tag{4}$$

Here the normalization has been chosen so as to be in agreement with Madison convention [20] and

$$\tau_q^{k^\dagger} = (-1)^q\tau_{-q}^k\,,$$

The matrix elements of the tensor operator are given by

$$\langle jm'|\tau_q^k(\vec{J})|jm\rangle = \sqrt{2k+1}\ C(jkj;mqm') \tag{5}$$

where $C(jkj;mqm')$ are the Clebsch-Gordan coefficients.

III. POSITIVE OPERATOR VALUED MEASURE

For a symmetric space, we define a set of POVMs as

$$E_q^k = \frac{\tau_q^k\tau_q^{k^\dagger}}{N^2}, \tag{6}$$

satisfying the properties of partition of unity, hermiticity and positivity.

A. To compute N

Since E_q^k add up to unity i.e., $\sum_{kq} E_q^k = 1$ in terms of matrix elements, we have

$$\langle jm| \sum_{kq} \frac{\tau_q^k \tau_q^{k^\dagger}}{N^2} |jm'\rangle = \delta_{mm'} \tag{7}$$

Consider

$$\begin{aligned}\langle jm| \sum_{kq} \tau_q^k \tau_q^{k^\dagger} |jm'\rangle &= \sum_{m_1 kq} (-1)^q \langle jm|\tau_q^k|jm_1\rangle\langle jm_1|\tau_{-q}^k|jm'\rangle \\ &= \sum_{m_1 k' q'} (-1)^q [k']^2 C(jk'j; m_1 q' m) C(jk'j; m' q m_1)\end{aligned} \tag{8}$$

By using the symmetry properties of Clebsch-Gordan coefficients, we have

$$\begin{aligned}\langle jm| \sum_{kq} \tau_q^k \tau_q^{k^\dagger} |jm'\rangle &= \sum_{m_1 k' q'} [k']^2 C(jk'j; m_1 q' m) C(jk'j; m_1 q' m') \\ &= \sum_{k' q' m} [k']^2 C(jk'j; m_1 q' m)^2 \delta_{mm'}\end{aligned} \tag{9}$$

Therefore

$$\langle jm| \sum_{kq} \frac{\tau_q^k \tau_q^{k^\dagger}}{N^2} |jm'\rangle = \delta_{mm'} \tag{10}$$

with

$$N^2 = \sum_{k'q'} [k']^2 C(jk'j; m - q'\, q'm)^2 \tag{11}$$

B. Hermiticity

$E_q^{k^\dagger} = E_q^k$ for all k, q

$$(\tau_q^k \tau_q^{k^\dagger})^\dagger = (\tau_q^{k^\dagger})^\dagger \tau_q^{k^\dagger} = \tau_q^k \tau_q^{k^\dagger} \tag{12}$$

C. Positivity

$\langle\psi|E_q^k|\psi\rangle \geq 0$ for all k, q and all $\langle\psi| \in H$, i.e., E_q^k are positive operators.

Observe that the POVMs do not form an orthogonal set. If we perform the measurement and do not record the results then the post measurement state is described by the density operator,

$$\rho^f = \sum_{kq} \frac{E_q^k \rho^i E_q^k}{Tr(\rho^i E_q^k)} \tag{13}$$

IV. NEUMARK'S THEOREM

Next, we address the question, given the set of operators E_q^k acting on the symmetric space such that $\sum_{kq} E_q^k E_q^k = I$, can this be interpreted as resulting from a measurement on a larger space? The answer is yes. The details will be discussed at the time of presentation.

---

[1] S. E. Ahnert and M. C. Payne, Phys. Rev. A **71**, 012330 (2005).

[2] M. Ziman and V. Buzek, Phys. Rev. A **72**, 022343 (2005).

[3] S. E. Ahnert and M. C. Payne, Phys. Rev. A **73**, 022333 (2006).

[4] I.D. Ivanovic, Phys. Lett. A **123**, 257 (1987).

[5] D. Dieks, Phys. Lett. A **126**, 303 (1988).

[6] T. Mor, TelePOVM - New faces of teleportation, arXiv:quant-th/9608005v1,1996.

[7] T. Mor, P. Horodecki, Teleportation via generalized measurements, and conclusive teleportation, arXiv:quant-ph/9906039, 1999.

[8] F.L. Yan and H.W. Ding, Chin. Phys. Lett. 23 (2006) 17.

[9] J. Wu, Int. J. Theor. Phys. 49, 324 (2010).

[10] Z.Y. Wang, Int. J. Theor. Phys. 49, 1357 (2010).

[11] J.F. Song and Z.Y. Wang, Int. J. Theor. Phys. 50, 2410 (2011).

[12] P. Zhou, J. Phys. A 45, 215305 (2012).

[13] Siendong Huang, Physics Letters A 377, 448-451 (2013) and the references therein.

[14] Ram A. Somaraju, Alain Sarlette and Hugo Thienpont arXiv:1303.2631v2 [quant-ph] 12 Apr 2013

[15] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P.Walther, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. 103, 020503 (2009).

[16] Fano, U. National Bureau of Standards Report 1214 (1951)
Fano, U. Phys.Rev., **90**, 577 (1953).
Fano, U. Rev.Mod.Phys., **29**, 74(1957).
Fano, U. Rev.Mod.Phys., **55**, 855 (1983).

[17] A. Peres, Quantum Theory: Concepts and Methods (Kluwer Academic Publishers, Dordrecht, The Netherlands,1993).

[18] Racah, G. Group theory and spectroscopy. CERN, report 61-8 (1961).

[19] Rose, M. E. Elementary theory of angular momentum. John Wiley, New York (1957).

[20] Satchler et al. (1971). Proceedings of the International Conference on Polarization Phenomena in Nuclear Reactions. University of Wisconsin Press, Madison