Zero-error classical channel capacity and simulation cost assisted by quantum non-signalling correlations

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When a communication channel $\mathcal{N}$ from Alice (A) to Bob (B) can be used to simulate another channel $\mathcal{M}$ that is also from A to B? We can abstractly represent the simulation process as the FIG.1. This problem has many variants according to the resources available to A and B. In particular, the case when A and B can access unlimited amount of shared entanglement has been completely solved. Let $C_E(\mathcal{N})$ denote the entanglement-assisted classical capacity of $\mathcal{N}$ [1]. It was shown that, in the asymptotic setting, to optimally simulate $\mathcal{M}$, we need to apply $C_E(\mathcal{M})/C_E(\mathcal{N})$ times of $\mathcal{N}$ [2]. In other words, the entanglement-assisted classical capacity uniquely determines the property of the channel in the simulation process.

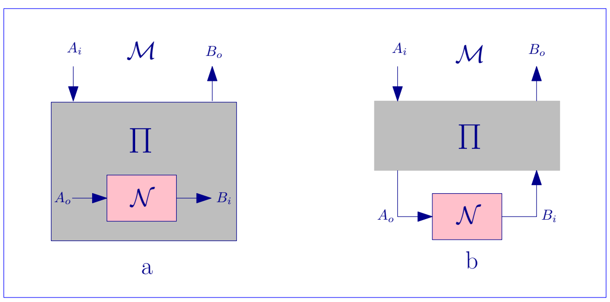

FIG. 1. A general simulation network: a). We have abstractly represented the general simulation procedure for implementing a channel $\mathcal{M}$ using another channel $\mathcal{N}$ just once, and the correlations between A and B; b). This is just an equivalent way to redraw a), and we have highlighted all correlations between A an B, and their pre- and/or post- processing as Π, a quantum non-signalling correlation.

We are interested in the zero-error case first studied by Shannon in 1956 [3]. It is well known that determining the zero-error classical capacity is generally extremely difficult even for classical channels. Remarkably, by allowing a feedback link from the receiver to the sender, Shannon proved that the zero-error classical capacity is given by an interesting quantity which was later called the fractional packing number. This number only depends on the bipartite graph induced by the classical channel under consideration, and has a simple linear programming characterization. Recently Cubitt *et al* introduced classical non-signalling correlations into the zero-error simulation problems for classical channels, and proved that the well-known fractional packing number gives precisely the zero-error classical capacity of the channel [4].

A class of quantum non-signalling correlations has been introduced as a natural generalization of classical non-signalling correlations [5] [6]. Any such correlation is described by a two-input and two-output quantum channel with non-signalling constraints between A and B (refer to $\Pi : \mathcal{L}(A_i \otimes B_i) \to \mathcal{L}(A_o \otimes B_o)$ in FIG.1). We imitate the approach in [4] to study the zero-error classical capacity of a general noisy quantum channels and the reverse problem of simulation, both assisted by this more general class of quantum non-signalling correlations. We show below that both problems can

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be completely solved in the one-shot scenarios, and the solutions are given by semi-definite programmings (SDPs). To describe these results, we need to introduce a few notations. Let $\mathcal{N}$ be a quantum channel with a Kraus operator sum representation $\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger$, where $\sum_k E_k^\dagger E_k = \mathbb{1}$. Let $K = \text{span}\{E_k\}$ denote the Kraus operator space of $\mathcal{N}$. The Choi-Jamiołkowski matrix of $\mathcal{N}$ is given by $J_{AB} = (\text{id}_A \otimes \mathcal{N})\Phi_{AA'}$ with $\Phi_{AA'}$ the unnormalized maximally entangled state. Let P_{AB} denote the projection on the support of J_{AB}.

The one-shot zero-error classical capacity of $\mathcal{N}$ assisted by quantum non-signalling correlations only depends on the Kraus operator space K, and is given by the integer part of following SDP

$$\begin{aligned} \Upsilon(K) = \max \operatorname{Tr} S_A \text{ s.t. } & 0 \le U_{AB} \le S_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}_A U_{AB} = \mathbb{1}_B, \\ & \operatorname{Tr} P_{AB}(S_A \otimes \mathbb{1}_B - U_{AB}) = 0. \end{aligned}$$

Similarly, the exact simulation problem has a SDP formulation. The one-shot zero-error classical cost of simulating a quantum channel $\mathcal{N}$ with Choi matrix J_{AB} is given by $\lceil 2^{-H_{\min}(A|B)_J} \rceil$ messages per channel realization, where $H_{\min}(A|B)_J$ is the conditional min-entropy defined as follows [7]:

$$2^{-H_{\min}(A|B)_J} = \min \operatorname{Tr} \Gamma_B, \text{ s.t., } J_{AB} \le \mathbb{1}_A \otimes \Gamma_B.$$

Since the conditional min-entropy is additive, it follows immediately that the asymptotic simulation cost of a channel is given by $-H_{\min}(A|B)_J$ bits per channel realization. As a direct consequence, the asymptotic zero-error classical simulation cost of the cq-channel $0 \to \rho_0$ and $1 \to \rho_1$, is given by $\log(1 + D(\rho_0, \rho_1))$, where $D(\rho_0, \rho_1) = ||\rho_0 - \rho_1||_1/2$ is the trace distance between ρ_0 and ρ_1. This provides a new operational interpretation of the trace distance between ρ_0 and ρ_1 as the asymptotic exact simulation cost for the above cq-channel.

Since there might be more than one channels with Kraus operator spaces included in K, we are interested in the exact simulation cost of the cheapest channel. The exact simulation cost $\Sigma(K)$ of the cheapest channel $\mathcal{N}$ such that $K(\mathcal{N}) < K$ (supporting on P_{AB}), is given by the integer part of

$$\begin{aligned} \Sigma(K) = \min \operatorname{Tr} T_B \text{ s.t. } & 0 \le V_{AB} \le \mathbb{1}_A \otimes T_B, \\ & \operatorname{Tr}_B V_{AB} = \mathbb{1}_A, \\ & \operatorname{Tr}(\mathbb{1} - P)V = 0. \end{aligned}$$

Let us now introduce the asymptotic zero-error channel capacity and simulation cost of K as follows,

$$C_{0,NS}(K) = \sup_{n \ge 1} \frac{\log \Upsilon(K^{\otimes n})}{n},$$
$$G_{0,NS}(K) = \inf_{n \ge 1} \frac{\log \Sigma(K^{\otimes n})}{n}.$$

In general, one-shot solutions do not give the asymptotic results, and feasible formulas for the asymptotic capacity and simulation cost remain unknown.

Nevertheless, for the case K corresponds to a cq-channel $\mathcal{N} : i \to \rho_i$, we show that the zero-error classical capacity is given by the solution of the following simplified SDP

$$\mathsf{A}(K) = \max \sum_i s_i, \text{ s.t. } 0 \le s_i, \sum_i s_i P_i \le \mathbb{1},$$

and P_i is the projection on the support of ρ_i. $\mathsf{A}(K)$ was introduced by A. Harrow as a natural generalization of the Shannon's classical fractional packing number [8], and can be named as *semidefinite (fractional) packing number* associated with a set of projections $\{P_i\}$. Then our result can be summarized as

$$C_{0,NS}(K) = \log \mathsf{A}(K).$$

This capacity formula naturally generalizes the result in [4], and has two interesting corollaries. First, it implies that the zero-error classical capacity of cq-channels assisted by quantum non-signalling correlations is additive, i.e.,

$$C_{0,\text{NS}}(K_0 \otimes K_1) = C_{0,\text{NS}}(K_0) + C_{0,\text{NS}}(K_1),$$

for any two Kraus operator spaces K_0 and K_1 corresponding to cq-channels.

Second, and more importantly, we show that for any undirected classical graph $G = (V, E)$ with vertices $V = \{1, ..., n\}$ and edges $E \subset$

$V \times V$, the Lovász ϑ function $\vartheta(G)$ [9], is an achievable lower bound of the zero-error classical capacity assisted by quantum non-signalling correlations of any quantum channel $\mathcal{N}$ that has G as its non-commutative graph in the sense of [11]. For simplicity, we denote the non-commutative graph generated by the graph G as

$$G = \operatorname{span}\{|i\rangle\langle j| : (i,j) \in E \text{ or } i = j, i, j \in V\}.$$

We also define the zero-error classical capacity of a graph G assisted by quantum non-signalling correlations as

$$C_{0,\mathrm{NS}}(G) = \min\{C_{0,NS}(K) : K^\dagger K = G\}.$$

Then we have

$$C_{0,\mathrm{NS}}(G) = \log \vartheta(G).$$

Thus the Lovász ϑ function of a graph G can be operationally interpreted as the zero-error classical capacity of the graph assisted by quantum non-signalling correlations. To the best of our knowledge, this is the first complete operational interpretation of the Lovász ϑ function since 1979. Previously it was shown that the Lovász ϑ function is an upper bound for the zero-error entanglement-assisted classical capacity of a graph [10][11]. It would be quite interesting to know whether the use of quantum non-signalling correlations could be replaced by shared entanglement.

Note that for a classical channel with bipartite graph Γ, such that

$$K = \operatorname{span}\{|j\rangle\langle i| : i \to j \text{ edge in } \Gamma\},$$

it was shown in [4] that

$$C_{0,\mathrm{NS}}(K) = G_{0,\mathrm{NS}}(K) = \log \mathsf{A}(K) = \log \alpha^*(\Gamma).$$

In fact, there it was shown that

$$\Upsilon(K) = \Sigma(K) = \mathsf{A}(K).$$

That is, in the presence of classical non-signalling correlations, the zero-error communication and simulation for a bipartite graph are reversible. However, this is not true even for a simple cq-channel $\mathcal{N}$ with two pure output states $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi_1\rangle = \alpha|0\rangle - \beta|1\rangle$, with $\alpha \geq \beta = \sqrt{1-\alpha^2}$. We can assume $\alpha > \beta > 0$ since the two equality cases are trivial. Note $|\langle\psi_0|\psi_1\rangle| = \alpha^2 - \beta^2 = 2\alpha^2 - 1$. We can work out all the SDPs introduced above:

$$\begin{aligned}
\Upsilon(K) &= 1,\\
\Upsilon(K \otimes K) &\geq \max\left\{1, \frac{1}{2\alpha^4}\right\},\\
\Upsilon(K^{\otimes n}) &\geq \frac{1}{\alpha^{2n} + \beta^{2n}} \quad \text{for } n \gg 1,\\
\mathsf{A}(K) &= \frac{1}{\alpha^2} = \frac{2}{1 + |\langle\psi_0|\psi_1\rangle|},\\
C_{\min,\mathrm{E}}(K) &= H(\alpha^2, \beta^2),\\
2^{-H_{\min}(A|B)_J} &= 1 + \frac{1}{2}\|\psi_0 - \psi_1\|_1 = 1 + 2\alpha\beta = \Sigma(K),
\end{aligned}$$

where $C_{\min,\mathrm{E}}(K)$ is the minimum of the entanglement-assisted classical capacity of $\mathcal{N}$ such that $K(\mathcal{N}) < K$.

We get

$$\begin{aligned}
C_{0,\mathrm{NS}}(K) &= \log \mathsf{A}(K) = -2\log\alpha\\
C_{\min,\mathrm{E}}(K) &= H(\alpha^2, \beta^2),\\
G_{0,\mathrm{NS}}(K) &= \log(1 + 2\alpha\beta).
\end{aligned}$$

Clearly, we have

$$C_{0,\mathrm{NS}}(K) < C_{\min,\mathrm{E}}(K) < G_{0,\mathrm{NS}}(K),$$

for any $\alpha > \beta > 0$. This demonstrates the irreversibility of zero-error communication and simulation even for such a simple cq-channel.

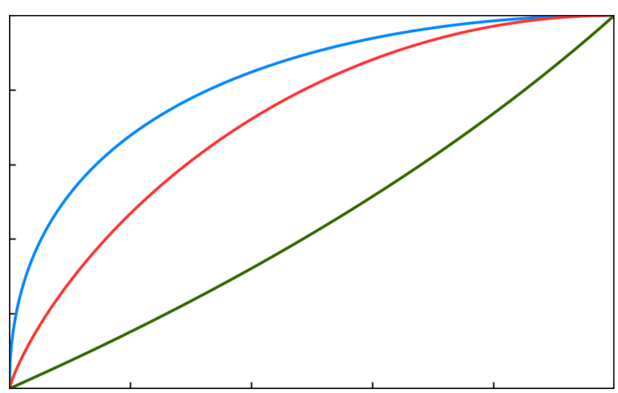

FIG. 2. Comparison between $C_{0,\mathrm{NS}}$ (green), $C_{\min\ E}$ (red) and $G_{0,\mathrm{NS}}$ (blue) for the cq-channel of two pure states, as a function of $0 \leq \beta^2 \leq \frac{1}{2}$.

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