Characterizing Genuine Multipartite Entanglement in 2D Quantum Spin Lattices

Himadri Shekhar Dhar[1], Aditi Sen(De)[2], and Ujjwal Sen[2]
[1]*School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India*
[2]*Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, India*

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Genuine multisite entanglement is an important resource in quantum information protocols and is known to offer significant advantage in quantum tasks in comparison to bipartite entanglement [1]. In particular, it is the basic ingredient in measurement-based quantum computation [2], and is beneficial in various quantum communication protocols [3]. Apart from the conventional information tasks, the study of multisite entanglement turns out to be important in understanding many-body phenomena like quantum phase transitions [4] and in biological mechanisms to understand transport properties in the evolution of photosynthetic complexes [5]. Although bipartite entanglement in the case of two spin-1/2 particles is rather well-understood, the situation is quite different in the case of the classification and quantification of entanglement in higher dimensions as well as in multiparty systems. The fact that many-particle systems can have different types of useful entanglement, depending on the particular information processing protocols under study, makes the characterization and quantification of entanglement in multiparty systems a difficult proposition.

In this work, we consider quantum states, with dimer coverings, of 2D quantum spin-1/2 lattice in the form even and odd spin ladders and isotropic square lattices (see Fig. 1). These quantum spin-1/2 lattice states, with short-range dimer coverings, are possible ground states of quantum spin liquids, known as resonating valence bond (RVB) states [6], and are of considerable interest in high-temperature superconductivity [7], cooperative phenomena in many body systems [8],and quantum computation [9]. These short-range dimer states can be efficiently simulated in laboratories using atoms in optical lattices and cavities using interacting photons [10]. An interesting aspect of quantum spin-1/2 lattice states, with short-range dimer coverings, is the relation between its entanglement properties and the geometry of the state [11]. A striking feature of the quantum spin ladder is that the interpolation from the 1D spin chain to the 2D square lattice by gradually increasing the number of legs is not straightforward. For example, the quantum characteristics of the Heisenberg ladder ensure that the odd- and the even-legged ladder ground states have different correlation properties. Thus, the scaling of entanglement properties of even and odd 2D spin ladders to the isotropic square 2D spin lattice is fundamentally as well as technologically important.

It can be shown using purely symmetry properties and information-theoretic arguments, that 2D isotropic spin-1/2 rotationally invariant states are always genuinely multisite entangled, irrespective of the lattice geometry and dimension. However, the large number of superpositions of nearest-neighbor (NN) spin-1/2 dimer coverings even in moderately large 2D ladders and square lattices, makes the analytical characterization and computation of both bipartite and multisite entanglement an arduous task. In this work, we introduce an analytical iterative method, the *density matrix recursion method* (DMRM) [12], to generate arbitrary reduced density matrices of superpositions of short-range dimer coverings on periodic or non-periodic quantum spin-1/2 lattices, for the ladder and square configuration. The DMRM technique allows us to recursively express density matrices of large spin-1/2 2D lattice in terms of smaller lattices. This reduces the computational complexity involved in the characterization of large sized spin systems. We analytically calculate the exact genuine multipartite entanglement, using a computable geometric measure of multisite entanglement, the *generalized geometric measure* (GGM) [13], for the considered quantum spin-1/2 states with short-ranged dimer coverings, on ladder and square lattice with arbitrary number of sites.

We observe, in Fig. 2, that the characterization of genuine multisite entanglement in the states of both odd- and even-legged ladders can be obtained by using the DMRM method, for relatively

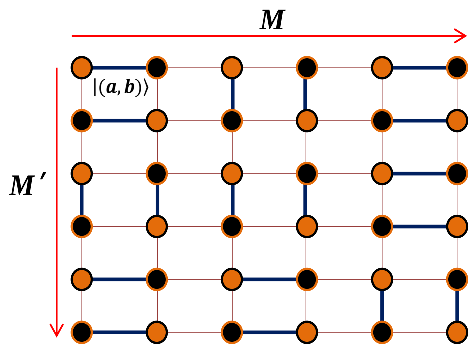

Figure 1: A 2D quantum spin-1/2 system, with NN dimer coverings, can be presented in the form of a bipartite lattice. The sublattices A and B are denoted by light and dark circles, repectively. The thick solid lines show the NN dimer states ($|(a_i, b_j)\rangle$) from a site in sublattice A to another in B. The figure represents a possible dimer covering. The final state is the superposition of all such dimer coverings. The ladder states correspond to $M' < M$. For, the square lattice $M' = M$. In general, such superposed quantum spin-1/2 states can be expressed in terms of the Rokhsar-Kivelson Hamiltonians [14].

larger system sizes than that is possible by exact diagonalization. We find that the scaling of genuine multisite entanglement can capture the disparity between the even and odd ladder states. Hence, we observe that the geometry of spin ladders play an important role in the scaling of multipartite entanglement [12]. We have shown that the DMRM technique can also be obtained for the isotropic square spin-1/2 lattice. The GGM values of square and approximate square spin-1/2 lattices is shown in Fig. 3 [15]. The results show that the finite values of odd and even ladders can be scaled to the isotropic lattice. Using finite-size scaling, we can estimate the scaling of multisite entanglement to infinite 2D spin-1/2 lattices, where the odd and even disparity is removed.

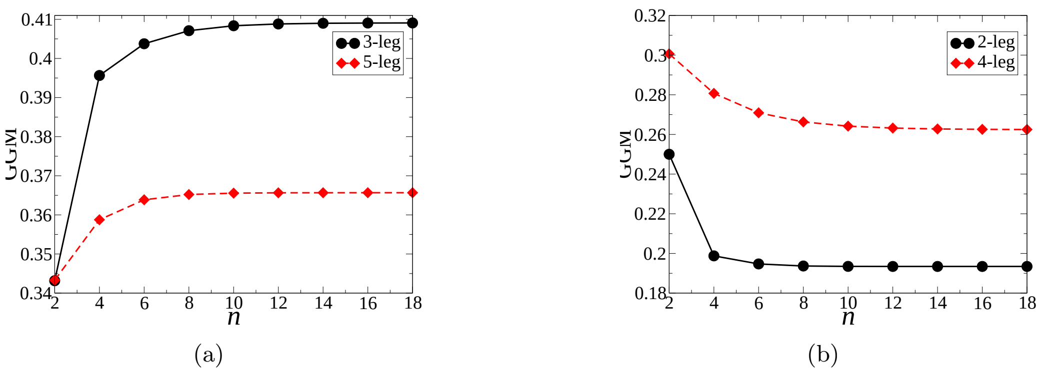

Figure 2: (a) Genuine multisite entanglement (a) increases with system size for odd ladders, and (b) decreases for even ladders. The vertical axis represents the GGM, while the horizontal one represents the number of rungs. The DMRM technique thus allows us to calculate multisite entanglement properties for ladders with as many as 90 spins for 5-legged ladder. Both axes are dimensionless.

To summarize, the characterization and calculation of multisite entanglement in large spin lattices is a conceptually and computationally difficult task. In this work, we investigate multipartite entanglement of large superposed state consisting of dimer coverings of a quantum spin-1/2 states, for ladder and square lattices. We show that the multisite entanglement distinguishes between even and odd ladders which scale differently with size. Extending the analysis to square 2D lattices, we observe, using finite-size scaling that infinite size lattices are finitely multisite entangled. Hence, at the infinite limit both odd and even ladders converge to the square 2D limit.

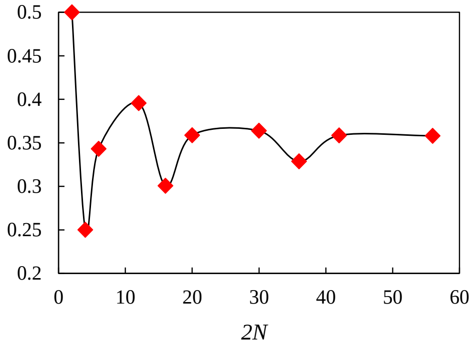

Figure 3: The behavior of GGM in the case of a square spin-1/2 lattice, with increasing total number of spins (2N). We use finite size scaling to estimate the GGM for infinite square lattices. The scaling analysis gives $G(|\psi\rangle) \approx G_c(|\psi\rangle) \pm k\ \mathbb{N}^{-x}$ where $\mathbb{N} = 2N$ and $G_c(|\psi\rangle)$ is an estimated value of GGM for the infinite lattice, based on the average of values of GGM given in Fig.1(b), with k being a constant. The value of x, estimated by finite-size scaling, using $G_c(|\phi_N\rangle)$=0.358, is x =1.69, while k=0.843.

Genuine multisite entanglement is potentially a basic ingredient in building large-scale quantum computers and also in implementation of multiparty quantum communication. The method presented can be a useful tool if such highly superposed systems are considered for performing quantum tasks. Specifically, the iterative method can be employed to derive reduced density matrices that will in turn be fruitful in the calculation of nearest-neighbor bipartite entanglement as well as that of other two-point correlation functions over and above the multisite properties that are calculated in our works.

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