A minimal state-dependent proof of measurement contextuality for a qubit

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The doubts of EPR [1] and the theorems of Bell, Kochen and Specker [2, 3, 4] established the impossibility of some very natural hidden variable models of quantum theory, characterized by locality and noncontextuality. Spekkens recently generalized this class of models to generalized-noncontextual ontological models, defining them in a manner that is applicable to any operational theory rather than quantum theory alone [5]. The central motivation of these models is to illuminate the distinction between the classical and the quantum world in a manner that is mathematically sound and, hopefully, experimentally testable, e.g., through violations of Bell inequalities.

Our main result is a demonstration that quantum correlations do better than classical correlations in a particularly simple scenario involving a qubit and three unsharp two-valued quantum measurements [6]. In particular, we consider a noncontextual inequality obtained by Liang et al.[7] based on a version of contextuality that was first discussed by Specker in 1960 [8], well before the Bell-Kochen-Specker theorem. This inequality concerns the stength of anti-correlations in the simplest conceivable contextuality scenario which we refer to as Specker's scenario. We also refer to the inequality as Specker's inequality. A contextuality scenario is a collection of subsets, called 'contexts', of the set of all measurements being considered. A context refers to measurements that can be jointly implemented. Specker's scenario requires three two-valued measurements, $\{M_1, M_2, M_3\}$, to allow for three non-trivial contexts: $\{\{M_1, M_2\}, \{M_1, M_3\}, \{M_2, M_3\}\}$. Each measurement takes values in $\{+1, -1\}$.

In Specker's scenario, measurement statistics that always shows perfect anti-correlation between any two measurements sharing a context is necessarily contextual. On assigning outcomes $\{+1, -1\}$ noncontextually to the three measurements $\{M_1, M_2, M_3\}$, it becomes obvious that the maximum number of anti-correlated contexts possible in a single assignment is two, e.g., for the assignment $\{M_1 \to +1, M_2 \to -1, M_3 \to +1\}$, $\{M_1, M_2\}$ and $\{M_2, M_3\}$ are anti-correlated but $\{M_1, M_3\}$ is not. This puts an upper bound of $\frac{2}{3}$ on the probability of anti-correlation when a context is chosen uniformly at random. Specker's scenario precludes projective measurements because a set of three pairwise commuting projective measurements is trivially jointly measurable and cannot show contextuality. One may surmise that it represents a kind of contextuality that is not seen in quantum theory. However, as Liang et al. [7] showed, this contextuality scenario can be realized using noisy spin-1/2 observables. They showed that if one does not assume outcome determinism for unsharp measurements and models them stochastically but noncontextually, then this generalized-noncontextual model [5] for noisy spin-1/2 observables will obey a bound of $1 - \frac{\eta}{3}$, where $\eta \in [0, 1]$ is the sharpness associated with each observable. Formally,

$$R_3 \equiv \frac{1}{3} \sum_{(ij)\in\{(12),(23),(13)\}} \Pr(M_i \neq M_j) \leq 1 - \frac{\eta}{3}, \tag{1}$$

where $\Pr(M_i \neq M_j)$ is the probability of anti-correlation between measurements M_i and M_j. After giving examples of orthogonal and trine spin-axes that did not seem to show a violation of this inequality, Liang et al. [7] left open the question of whether such a violation exists. They conjectured that all such triples of POVMs will admit a generalized-noncontextual model [5], i.e., Specker's inequality (1) will not be violated.

We show that this is not the case [6]. In particular, we deal with triples of unsharp qubit POVMs in full generality, rather than only considering special cases like orthogonal and trine spin axes, and show that while they do not admit a state-independent violation of Specker's inequality, they do allow a state-dependent violation. We also obtain the optimal state-dependent violation of this inequality. This result is important from the point of view of minimal proofs of contextuality, the interplay of joint measurability and contextuality of quantum theory, and the fact that even without entanglement there is something nonclassical about corrrelations between single qubit measurement outcomes.

Contextuality precludes the possibility that quantum measurements reveal pre-existing outcomes. It arises from the non-existence of a global joint probability distribution over measurement outcomes that can reproduce the measurement statistics predicted by quantum theory. Traditionally, contextuality has been shown with respect to noncontextual hidden variable models of projective measurements for Hilbert spaces of dimension three or greater [4, 9, 10]. While a state-independent proof of contextuality holds for any state-preparation, a state-dependent proof requires a special choice of the prepared state. The minimal state-independent proof of traditional contextuality requires a qutrit and 13 projectors [10, 11]. The minimal state-dependent proof [9, 12], first given by Klyachko et al. [9], requires a qutrit and five projectors. What we have shown in Ref. [6] is that a simpler contextuality scenario, viz. Specker's, is realizable in quantum theory if one moves beyond projective measurements and considers the possibilities allowed by qubit POVMs. This allows us to give a state-dependent proof of generalized contextuality using a qubit and three unsharp measurements. Our proof is minimal in this sense.

Besides, our result hints at the fact that perhaps all contextuality scenarios may be realizable and contextuality for these may be demonstrated in quantum theory if we consider the possibilities that general quantum measurements allow. In particular, scenarios that involve pairwise compatibility between all measurements but no global compatibility may be realizable within quantum theory. Specker's scenario is the simplest such example we have considered.

To summarize, the joint measurability allowed in a theory restricts the kind of contextuality scenarios that can arise in it. Quantum theory admits Specker's contextuality scenario if one uses unsharp measurements [7]. As we have shown, quantum theory also allows violation of the noncontextual bound for anti-correlations in this scenario. Thus, quantum theory is contextual even in the simplest contextuality scenario, a fact that wasn't known before our work [6].

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