Title: Local Unitary Equivalent Classes of Symmetric N-Qubit Mixed States

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Abstract: We develop a method of classifying LU equivalent classes of symmetric N-qubit mixed states based on multiaxial representation [1] of the density matrix. Bastin et.al [2] have defined two parameters, diversity degree and degeneracy configuration, to characterize symmetric N-qubit pure states using Majorana construction. In our scheme of classification, in addition to these two parameters, we introduce another parameter called the rank . The power of our method is demonstrated using several well known examples of symmetric two, three qubit pure states and two qubit mixed states. A recipe to identify the most general symmetric N-qubit pure separable state is also given.

Introduction:

Local unitary (LU) equivalence of multipartite pure states has received a lot of attention recently [3-6]. ρ and ρ' are said to be LU equivalent if $\rho' = U\rho U^{\dagger}$ where $U \in SU(2)^{\times N}$. Local Operation and Classical Communication (LOCC) equivalence classes are defined such that all quantum states within the same class can be transformed to each other by LU transformation [3]. It is well known that the states belonging to the same LU equivalent class can be used for similar kind of quantum information processing tasks as they possess the same amount of entanglement. One way of classifying these states is by evaluating the LU invariants. Well known algebraic methods for generation of invariants already exist in literature [7-10]. As the number of subsystems increases, the problem of identifying and interpreting the independent invariants rapidly becomes very complicated. However, LU invariants associated with the symmetric states, which are experimentally viable and mathematically elegant, are easier to handle as the dimensionality of the Hilbert space involved is much less. Because of the permutational symmetry involved in the symmetric state, ρ and ρ' are said to belong to the same LU equivalent class if $\rho' = R \otimes R \otimes R \ldots R\rho R^{-1} \otimes R^{-1} \ldots R^{-1}$ where R represents the rotation operator on a qubit [11]. LU invariants of the most general symmetric mixed systems have been constructed using the elegant multiaxial representation of the symmetric states [12]. Different LUequivalent classes of up to 5 pure qubit states and for certain mixed states have been determined by introducing the standard form for multipartite states [4, 5]. Entanglement classification of mixed state under LU transformation poses a difficult problem as the definition of mixed state entanglement itself is poorly understood. However, an operational entanglement classification of symmetric mixed state for an arbitrary number of qubits under Stochastic Local Operation and Classical Communication (SLOCC) has been introduced by Bastin et. al[13]. In this paper we propose a scheme for classifying the most general symmetric N-qubit mixed states under LU transformation based on the multi-axial representation of Ramachandran and Ravishankar [1].

Set of N-qubit pure states that remain unchanged by permutations of individual particles are called symmetric states. Symmetric states offer elegant mathematical analysis as the dimension of the Hilbert space reduces drastically from 2^N to (N+1), when N-qubits respect exchange symmetry. Such a Hilbert space is considered to be spanned by the eigen states $\{|j,m\rangle;\ -j \leq m \leq +j\}$ of angular momentum operators $J^2$and J_z, where $j = \frac{N}{2}$. Fortunately, a large number of experimentally relevant states [14] possesses symmetry under particle exchange and this property allows us to significantly reduce the computational complexity.

Multi-axial Representation of Density matrix:

The standard expression for the most general spin-j density matrix can be written in terms of Fano statistical tensor parameters $t_q^k{}'s$ (k=0,1,…2j and q= -k…..+k). It has been shown by Ramachandran and Ravishankar[1] that any spherical tensor of rank k can be represented geometrically by a set of k vectors $\hat{Q}_i$ on the surface of a sphere of radius r_i. As the state of spin-j assembly is characterized by 2j spherical tensors, the state can be represented geometrically by a set of 2j spheres, one corresponding to each value of k, the kth sphere having k vectors specified on its surface. Thus, the spin-j system is in general characterized by j(2j+1) axes and 2j scalars. Since scalar product between $\hat{Q}(\theta_i, \varphi_i)$ and $\hat{Q}(\theta_j, \varphi_j)$ is an invariant under rotation, there are $\binom{j(2j+1)}{j} + 2j$ invariants [12].

LU Classification:

Coming to the entanglement classification of symmetric N-qubit pure state $|\psi\rangle = \mathcal{N} \sum_{1\leq i_1 \neq \cdots \neq i_N \leq N} |\epsilon_{i_1}, \ldots, \epsilon_{i_N}\rangle$ where $\mathcal{N}$ is a normalization factor and the $|\epsilon_i\rangle's$ s are single qubit states $\alpha_i|1\rangle + \beta_i|0\rangle$ with $|\alpha_i|^2 + |\beta_i|^2$=1, Bastin et. al, [2] introduce two parameters diversity degree (d) and degeneracy configuration ($D_{\{n_i\}}$). For example, a symmetric N-qubit state with all ϵ_i identical has a degeneracy configuration D_N and a diversity degree d of 1. If all but one ϵ_i are identical, we get the configuration $D_{N-1,1}$and $d = 2$. If all but two ϵ_i are identical, we get the configuration $D_{N-2,2}$ $(d = 2)$ or $D_{N-2,1,1}$ $(d = 3)$, depending on whether the two remaining ones are identical or not, respectively. We generalize this to symmetric N-qubit mixed state and define these two parameters for the spherical tensor parameters t^k_q. In addition to this, we define another parameter called the rank k which refers to the rank of the spherical tensor parameter t^k. Thus the notation becomes $D^k_{n_i}$. For example, in the case of a symmetric two qubit mixed state, the density matrix is characterized by t^1_q and t^2_q. In the case of t^1, there is only one axis. Thus we have only D^1_1. In the case of t^2, there are two axes in general. Hence the configurations are, D^2_2 and $D^2_{1,1}$. Thus in a two qubit system there are three different degeneracy configurations: D^1_1, D^2_2 and $D^2_{1,1}$. Similarly, a symmetric three qubit mixed system is characterized by t^1, t^2 and t^3. Thus the degeneracy configurations are: $\{D^1_1\}, \{D^2_2, D^2_{1,1}\}, \{D^3_3, D^3_{2,1}, D^3_{1,1,1}\}$.

Employing the above classification scheme, a recipe for identifying N-qubit pure separable state is discussed in detail. Some well known examples of symmetric two and three qubit pure states are also investigated. Classification of uniaxial, biaxial and triaxial mixed states which can be produced in the laboratory is illustrated.

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