Hardy's nonlocality argument as a witness for postquantum correlations

Subhadipa Das,[1,*] Manik Banik,[2,†] Ashutosh Rai,[1,‡] MD Rajjak Gazi,[2,§] and Samir Kunkri[3,∥]
[1]*S. N. Bose National Center for Basic Sciences, Block JD, Sector III, Salt Lake, Kolkata-700098, India*
[2]*Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata-700108, India*
[3]*Mahadevananda Mahavidyalaya, Monirampore, Barrakpore, North 24 Parganas-700120, India*
(Received 19 September 2012; published 14 January 2013)

Recently, Gallego *et al.* [Phys. Rev. Lett. **107**, 210403 (2011)] proved that any future information principle aiming at distinguishing between quantum and postquantum correlation must be intrinsically multipartite in nature. We establish similar results by using the device-independent success probability of Hardy's nonlocality argument for tripartite quantum systems. We construct an example of a tripartite Hardy correlation which is postquantum but satisfies not only the all-bipartite information principle but also the guess-your-neighbor's-input (GYNI) inequality.

DOI: 10.1103/PhysRevA.87.012112 PACS number(s): 03.65.Ud

I. INTRODUCTION

Recently, understanding the correlations among distant observers which are compatible with our current description of nature based on quantum mechanics has generated much interest. It has been shown that some general physical principles can restrict the set of no-signaling correlations among distant observers to a significant degree. Information theoretic principles like information causality [1] and nontrivial communication complexity [2,3] are novel proposals to single out the quantum correlations from the rest of the no-signaling correlations when two distant observers (or in some cases, even when more than two observers [5]) are involved. However, by applying the known information principles to the bipartite case, it has not been possible to derive the full set of quantum correlations resulting from the Hilbert space structure of quantum mechanics.

For a multipartite (more than two subsystems) scenario, the situation becomes even more complex and extremely challenging. Very recently, some interesting results [4,5] have been produced when more than two distant observers are involved. In [4], Gallego *et al.* provide an example of the tripartite no-signaling correlation, which is *time-ordered-bilocal* (TOBL) [6–8] and therefore respects any bipartite information principle, yet this correlation is nonquantum (unphysical) since it violates the *guess-your-neighbor's-input* (GYNI) inequality [9]. Thus this result demonstrates that in general any biprinciple is insufficient for deriving the set of all multiparty quantum (physical) correlations. A similar example is also provided in the work by Yang *et al.* [5], where an extremal point of the tripartite nonsignaling polytope is proved to be nonquantum. The nonquantum nature of this correlation is shown through violation of an inequality (Eq. (A1) in [5]), which is satisfied by all quantum correlations; however, in contrast with [4], this example respects the GYNI inequality.

In the present work we give a TOBL correlation which is nonquantum since it exceeds the maximum success probability of Hardy argument for tripartite quantum correlations. To show this, first we prove that in quantum mechanics the maximum success probability of the Hardy argument for a tripartite system is 0.125; earlier this result was known to hold only for projective measurement on three-qubits systems [10]. Then we explicitly construct a tripartite correlation in a general probabilistic theory with the following properties: (i) the correlation is TOBL and hence satisfies all bipartite information principles, and (ii) the correlation shows Hardy nonlocality with the success probability taking the value 0.2, which is greater than the maximum value 0.125 that can be achieved within quantum mechanics. Interestingly, the example we provide respects all known GYNI inequalities [9] and hence stands as good candidate to rule out GYNI as the principle which distinguishes quantum from postquantum.

The paper is organized as follows. In Sec. II we prove that the device-independent success probability for Hardy nonlocality for tripartite quantum system is $\frac{1}{8}$. In Sec. III we briefly described the TOBL correlations. In Sec. IV we present a tripartite no-signaling probability distribution which belongs to the TOBL set with a success probability for the Hardy nonlocality argument greater than that for quantum systems. Finally, in Sec. V we give our conclusions.

II. TRIPARTITE HARDY'S NONLOCALITY

Lucian Hardy first provided a simple argument for a bipartite system which reveals nonlocality within quantum mechanics without using any inequality [11]. Hardy's original argument can also be extended to multipartite systems. Hardy's test for tripartite systems in general probabilistic theories can be provided by considering the set of tripartite no-signaling correlations with binary input and binary output for each party—the set of such correlations is known to be points of a polytope in a 26-dimensional space with 53 856 extremal points [6]. A tripartite two-input–two-output Hardy correlation is defined by restricting a certain choice of 5 out of 64 joint probabilities in the correlation matrix [12–15]. Let $P(abc|xyz)$ denote the probability of obtaining output $a,b,c \in \{+1,-1\}$ conditioned that inputs of the three parties

*sbhdpa.das@bose.res.in
†manik11ju@gmail.com
‡arai@bose.res.in
§rajjakgazimath@gmail.com
∥skunkri@yahoo.com

are, respectively, $x \in \{X_0, X_1\}$, $y \in \{Y_0, Y_1\}$, and $z \in \{Z_0, Z_1\}$. The following five conditions, for example, define a tripartite Hardy correlation:

$$\begin{aligned} &P(+++|X_0Y_0Z_0) > 0, \\ &P(+++|X_1Y_0Z_0) = 0, \\ &P(+++|X_0Y_1Z_0) = 0, \\ &P(+++|X_0Y_0Z_1) = 0, \\ &P(---|X_1Y_1Z_1) = 0, \end{aligned} \tag{1}$$

where $\{X_0, X_1\}$, $\{Y_0, Y_1\}$, and $\{Z_0, Z_1\}$ are respective local observables corresponding to measurements performed by three distant parties, say Alice, Bob, and Charlie, and ± 1 are the possible measurement outcomes. One can easily show that the above correlation cannot be reproduced by any local realistic model.

The joint probability appearing in the first condition, inequality (1), is the success probability for Hardy's nonlocality argument. In quantum mechanics, for three-qubit systems subjected to local projective measurements, the maximum value of the success probability of the Hardy argument has been shown to be 0.125 [10]. In view of a result first conjectured in [16] and recently proved in [17], providing a device-independent maximum value for the success probability of a bipartite Hardy argument, one can ask what is the maximum probability of success for Hardy nonlocality for tripartite quantum systems of arbitrary local state space dimensions. In the following proposition, we show that the result (device independent bound) in the bipartite case can also be extended to the tripartite Hardy test [18]:

Proposition. The maximum value of the success probability of the Hardy argument for tripartite quantum systems is $\frac{1}{8}$.

Proof. As X_0 and X_1 are two Hermitian operators with eigenvalues ± 1 acting on a Hilbert space $\mathcal{H}$, thus we write

$$\begin{aligned} X_0 &= \Pi_{+|X_0} - \Pi_{-|X_0}, \\ X_1 &= \Pi_{+|X_1} - \Pi_{-|X_1}, \end{aligned}$$

where $\Pi_{a|x}$ are projection operators for $a \in \{+,-\}$ and $x \in \{X_0, X_1\}$. Then according to *Lemma 1*, stated in [17] and proved in [19], the Hilbert space $\mathcal{H}$ can be decomposed as a direct sum of subspaces $\mathcal{H}^i$ of dimension at most 2, such that $X_0 = \bigoplus_i X_0^i$ and $X_1 = \bigoplus_i X_1^i$, where X_0^i and X_1^i act on $\mathcal{H}^i$. Then we have $\Pi_{a|x} = \bigoplus_i \Pi^i_{a|x}$, where $\Pi^i_{a|x}$ acts on subspaces $\mathcal{H}^i$ for all a and x. We denote $\Pi^i = \Pi^i_{+|x} + \Pi^i_{-|x}$ the projector on $\mathcal{H}^i$. Obviously, same is also true at Bob's and Charlie's ends with similar notation used for Bob and Charlie.

In quantum mechanics, joint probabilities for the outcomes of measurements performed on three spacelike separated parts of a quantum system are given by

$$P(abc|xyz) = \mathrm{Tr}(\rho \Pi_{a|x} \otimes \Pi_{b|y} \otimes \Pi_{c|z}), \tag{2}$$

where ρ is the state of the system and $\Pi_{a|x}$, $\Pi_{b|y}$, and $\Pi_{c|z}$ are the measurement operators associated to outcomes a,b,c of measurements x,y,z, respectively. In general, the measurement operators are a positive operator-valued measure (POVM). As the dimension of the Hilbert space is not restricted, using Neumark's theorem we can consider measurement operators as projectors, without loss of generality. Therefore we can write

$$\begin{aligned} P(abc|xyz) &= \sum_{i,j,k} q_{ijk} \mathrm{Tr}\big(\rho_{ijk} \Pi^i_{a|x} \otimes \Pi^j_{b|y} \otimes \Pi^k_{c|z}\big) \\ &= \sum_{i,j,k} q_{ijk} P_{ijk}(abc|xyz), \end{aligned} \tag{3}$$

where $q_{ijk} = \mathrm{Tr}(\rho \Pi^i \otimes \Pi^j \otimes \Pi^k)$ and $\rho_{ijk} = (\Pi^i \otimes \Pi^j \otimes \Pi^k \rho \Pi^i \otimes \Pi^j \otimes \Pi^k)/q_{ijk}$ is, at most, a three-qubit state; $q_{ijk} \geqslant 0$ for all i,j,k and $\sum_{i,j,k} q_{ijk} = 1$. The Hardy's conditions for tripartite systems [i.e., condition (1)] are satisfied for P if and only if they are satisfied for each of the P_{ijk}, but then

$$P(+++|X_0Y_0Z_0) = \sum_{i,j,k} q_{ijk} P_{ijk}(+++|X_0Y_0Z_0) \tag{4}$$

is a convex sum of Hardy's probabilities in each three-qubit subspace. Being a convex sum, the success probability of Hardy's argument is therefore less or equal to the largest value in the combination, which is $\frac{1}{8}$ [10]. ■

III. TIME-ORDERED BILOCAL CORRELATIONS

A tripartite no-signaling probability distribution $P(abc|xyz)$ belongs to TOBL correlations [6–8] if it can be written as

$$P(abc|xyz) = \sum_\lambda p_\lambda P(a|x,\lambda) P_{B\to C}(bc|yz,\lambda) \tag{5}$$

$$= \sum_\lambda p_\lambda P(a|x,\lambda) P_{B\leftarrow C}(bc|yz,\lambda), \tag{6}$$

and analogously for $B|AC$ and $C|AB$, where p_λ is the distribution of some random variable λ, shared by the parties. The distributions $P_{B\to C}$ and $P_{B\leftarrow C}$ respect the conditions

$$P_{B\to C}(b|y,\lambda) = \sum_c P_{B\to C}(bc|yz,\lambda), \tag{7}$$

$$P_{B\leftarrow C}(c|z,\lambda) = \sum_b P_{B\leftarrow C}(bc|yz,\lambda). \tag{8}$$

From these equations it is clear that the distributions $P_{B\to C}$ allow signaling from Alice to Bob and $P_{B\leftarrow C}$ allow signaling from Bob to Alice. If a tripartite no-signaling probability distribution $P(abc|xyz)$ belongs to the set of TOBL distributions, all possible bipartite distributions derived by applying any local wirings on $P(abc|xyz)$ are local, i.e., the probability distribution $P(abc|xyz)$ respects all bipartite physical principles [7].

IV. NONQUANTUM CORRELATION SATISFYING THE BIPARTITE PRINCIPLE

Now we show that there exists a tripartite Hardy correlation which lies in the TOBL set. Since this correlation belongs to the TOBL set, it must satisfy all bipartite information principles. The probability distribution is given in Table I. In this table, $P(000|000) = \frac{1}{5}$, $P(000|001) = 0$, $P(000|100) = 0$, $P(000|010) = 0$, $P(111|111) = 0$. The success probability of Hardy argument for this correlation is $\frac{1}{5}$, which is strictly larger than the maximum

TABLE I. Tripartite no-signaling probability distribution $P(abc|xyz)$ with Hardy's success 1/5.

	abc							
xyz	000	001	010	011	100	101	110	111
000	$\frac{1}{5}$	0	0	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
001	0	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{5}$	0
010	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{10}$	0
011	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	0
100	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	0
101	0	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	0
110	0	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	0
111	$\frac{2}{5}$	0	0	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{1}{5}$	0

achievable quantum value $\frac{1}{8}$ and thus this is a nonquantum correlation.

To prove that the distribution $P(abc|xyz)$ belongs to the TOBL set, we show that it admits TOBL decomposition for all bipartition. Note that the correlation considered in Table I is symmetric under any permutation of the parties, so here it it is sufficient to provide a TOBL model in any one bipartition, say $A|BC$.

The probability distributions appearing in the TOBL decomposition for the bipartition $A|BC$ are such that for a given λ Alice's outcome a depends only her measurement settings x. Also, for given λ, $P_{B\to C}(b|y,\lambda)$ is independent of z, but for $B \to C$, c depends on both y and z. Similarly, for given λ, $P_{B\leftarrow C}(c|z,\lambda)$ is independent of y, but for $B \leftarrow C$, b depends on both y and z. Let a_x, b_y, and c_z denote the outcomes for Alice, Bob, and Charlie for the respective inputs x, y, and z. In Tables II and III, the outputs are deterministic and the weights p_λ are same. For any given λ, the outcome assignments of A in both tables are the same.

It is interesting to observe that the correlation in Table I satisfies the most general GYNI inequality [9] (see also [20]), whose violation certifies the nonquantum nature of a correlation. Thus the postquantum nature (which is guaranteed by the larger success probability for Hardy's argument compared to the quantum result) of the correlation given in Table I cannot be witnessed by violating the GYNI inequality, as happened for the correlation given in [4]. Therefore the nonlocality of

TABLE II. TOBL decomposition for the case $A|B \to C$.

λ	p_λ	a_0	a_1	b_0	b_1	c_{00}	c_{01}	c_{10}	c_{11}
1	$\frac{1}{10}$	0	0	1	1	1	0	1	1
2	$\frac{1}{10}$	0	0	1	1	1	1	0	1
3	$\frac{1}{10}$	1	0	0	0	1	1	1	0
4	$\frac{1}{10}$	1	0	1	0	0	0	1	0
5	$\frac{1}{5}$	1	0	1	0	1	0	1	0
6	$\frac{1}{10}$	0	1	0	0	0	1	1	1
7	$\frac{1}{10}$	0	1	0	1	0	1	0	0
8	$\frac{1}{10}$	1	1	1	0	0	0	0	1
9	$\frac{1}{10}$	1	1	0	1	1	0	0	0

TABLE III. TOBL decomposition for the case $A|B \leftarrow C$.

λ	p_λ	a_0	a_1	b_{00}	b_{01}	b_{10}	b_{11}	c_0	c_1
1	$\frac{1}{10}$	0	0	1	0	1	1	1	1
2	$\frac{1}{10}$	0	0	1	1	0	1	1	1
3	$\frac{1}{10}$	1	0	1	1	1	0	0	0
4	$\frac{1}{10}$	1	0	0	1	0	0	1	0
5	$\frac{1}{5}$	1	0	1	1	0	0	1	0
6	$\frac{1}{10}$	0	1	0	1	1	1	0	0
7	$\frac{1}{10}$	0	1	0	0	1	0	0	1
8	$\frac{1}{10}$	1	1	0	0	0	1	1	0
9	$\frac{1}{10}$	1	1	1	0	0	0	0	1

this postquantum correlation is qualitatively different from the nonlocality of the postquantum correlation that appeared in [4].

V. CONCLUSIONS

Distinguishing physically realized correlations from unphysical ones by some fundamental principle is an active area of research in the foundational perspective. Rather, it has been proved that nonlocality is a useful resource for device-independent cryptography [21]. So it is very important to know which nonlocal correlations can be obtained by physical means. Like the celebrated Bell inequality [22], the elegant argument of Hardy [11] reveals the nonlocality of quantum mechanics. Again, like the device independent value of the Bell violation for bipartite quantum mechanical systems, i.e., the Cirel'son bound [23], the optimal success probability for Hardy's nonlocality argument [17] in quantum mechanics could be a potential witness for detecting postquantum no-signaling correlations. We derive the device-independent success probability of Hardy's argument for tripartite quantum systems. Then we provide an explicit tripartite correlation which satisfies any bipartite information principles but shows Hardy nonlocality with a probability which is postquantum. In this way we establish that this device-independent value is a potential witness for tripartite postquantum correlations. Our example also satisfies most general GYNI inequalities. A newly proposed multipartite principle, namely local orthogonality (LO) [24], when applied to a single copy of tripartite no-signaling correlations is equivalent to the GYNI game. Of course, the witnessing power of the LO principle for detecting unphysical correlations increases when many copies of a correlation are considered, but the problem becomes computationally hard with an increasing number of copies. So it will be interesting to study multiple copies of the correlation provided in this work under the LO principle.

ACKNOWLEDGMENTS

It is a pleasure to thank Guruprasad Kar for motivating us in this direction. We also thank Sibasish Ghosh and Ramij Rahaman for many stimulating discussions regarding Hardy's nonlocality argument. S.D. and A.R. acknowledge support from the DST, Project No. SR/S2/PU-16/2007.

[1] M. Pawlowski, T. Paterek, D. Kaszlikowski, V. Scsrani, A. Winter, and M. Zukowski, Nature (London) **461**, 1101 (2009).

[2] W. van Dam, Ph.D. thesis, University of Oxford, 2000.

[3] Wim van Dam, arXiv:quant-ph/0501159.

[4] R. Gallego, L. E. Wurflinger, A. Acin, and M. Navascues, Phys. Rev. Lett. **107**, 210403 (2011).

[5] T. H. Yang, D. Cavalcanti, M. L. Almeida, C. Teo, and V. Scarani, New J. Phys. **14**, 013061 (2012).

[6] S. Pironio, J. D. Bancal, and V. Scarani, J. Phys. A: Math. Theor. **44**, 065303 (2011).

[7] R. Gallego, L. E. Wurflinger, A. Acin, and M. Navascues, Phys. Rev. Lett. **109**, 070401 (2012).

[8] J. Barrett, S. Pironio, J.-D. Bancal, and N. Gisin, arXiv:1112.2626v1 [quant-ph].

[9] M. L. Almeida, J. D. Bancal, N. Brunner, A. Acin, N. Gisin, and S. Pironio, Phys. Rev. Lett. **104**, 230404 (2010).

[10] S. K. Choudhary, S. Ghosh, G. Kar, S. Kunkri, R. Rahaman, and A. Roy, Quant. Inf. Comp. **10**, 0859 (2010).

[11] L. Hardy, Phys. Rev. Lett. **68**, 2981 (1992); **71**, 1665 (1993).

[12] G. Kar, Phys. Rev. A **56**, 1023 (1997).

[13] J. Cereceda, Phys. Lett. A **327**, 433 (2004).

[14] D. Boschi, S. Branca, F. De Martini, and L. Hardy, Phys. Rev. Lett. **79**, 2755 (1997).

[15] S. Kunkri and S. K. Choudhary, Phys. Rev. A **72**, 022348 (2005).

[16] K. Seshdreesan and S. Ghosh, J. Phys. A **44**, 315305 (2011).

[17] R. Rabelo, L. Y. Zhi, and V. Scarani, Phys. Rev. Lett. **109**, 180401 (2012).

[18] In fact, this result can be stated in a more general way. If an N-qubits system subjected to local projective measurements exhibit Hardy's nonlocality with the maximum success probability, say α, then for any N-partite quantum system subjected to general dichotomic local measurements, the maximum success probability of Hardy's nonlocality argument cannot exceed the value α. For $N = 2$ and $N = 3$, the respective values of α are 0.09 and $\frac{1}{8}$.

[19] L. Masanes, Phys. Rev. Lett. **97**, 050503 (2006).

[20] A. Acín, M. L. Almeida, R. Augusiak, and N. Brunner, arXiv:1205.3076v1 [quant-ph].

[21] A. Acin, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Phys. Rev. Lett. **98**, 230501 (2007).

[22] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy* (Cambridge University Press, Cambridge, UK, 2004).

[23] B. S. Cirel'son, Lett. Math. Phys. **4**, 93 (1980).

[24] T. Fritz, A. B. Sainz, R. Augusiak, J. B. Brask, R. Chaves, A. Leverrier, and A. Acín, arXiv:1210.3018v2 [quant-ph].