Entropic uncertainty relation for successive measurements

Kyunghyun Baek, Wonmin Son
Physics Department, Sogang University, Republic of Korea

Abstract

In quantum mechanics, the uncertainty is fundamental concept, so its underlying meaning has been argued intensively due to its weird properties. On this account, there has been various fomulations expressing the uncertainty principle in different ways. In this paper, the new entropic uncertainty relations will be suggested showing different aspects of uncertainty as we consider successive measurements. Already, Deutsch and Uffink suggested entropic uncertainty relations, so our work will be to modify it as focusing on another aspect (error-disurbance) of uncertainry.

- **The uncertainty relations**

Heisenberg is the first man who suggested the uncertainty principle. In 1927 [1], he proposed the uncertainty relation,

$$\epsilon(Q)\eta(P) \geq \frac{\hbar}{2} \tag{1}$$

where $\epsilon(Q)$ is the mean error occurring when an observer measures the position of a particle, and $\eta(Q)$ is the disturbance caused by the position measruement Q and $\hbar$ is Planks constatnt divided by 2π. The equation (1) shows that we can not measure position exactly without disturbing momentum.

The Heisenberg's relation (1) was later generalized to arbitrary pair of observables by Robertson [2]. Instead of position and momentum, he considered the generalized observables A and B. Additionally, the lower bound was also generalized to commutator of the observables, as follows.

$$\sigma(A)\sigma(B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \tag{2}$$

where $\sigma(Q)$ is standard deviation defined by $\sigma(Q)^2 = |\langle\psi|(Q - \langle Q\rangle)^2|\psi\rangle|$ and $[A,B]$ represents the commutator, $[A,B] = AB - BA$. This above relation (2) claims that in an arbitrary state $|\psi\rangle$ a pair of noncommuting observables can not be well localized simultaneously . In other words, it can be siad that we can not prepare well localized states simultaneously for noncommuting observables.

Afterwards Heisenberg's relation (1) was revised and improved in 2004 by M. Ozawa [7], which is called a universally valid error-disturbance uncertainty relation,

$$\epsilon(A)\eta(B) + \epsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle|. \tag{3}$$

In 2012 [3], it is proved that the old relation (1) is violated in spin measurements, but the improved one (3) is valid.

Consequently, nowdays we can summerize the interpretatioin of the uncertainty principle in three statements, [4]

(i) It is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized.
(ii) It is impossible to measure simultaneously position and momentum.
(iii) It is impossible to measure position without disturbing momentum.

In these statements, position and momentum represent conjugate variables.

According to the statements, we can classify the above relations. Firstly, the Robertson's (2) relation is equivalent with the statement (i), since from these relations we can conclude that there is a limitation of preparation of states in which noncommuting observables are well localized. Secondly, the error-disturbance relations (1)(3) are equivalent with the statement (iii), since they describe a situation that a measurement for observable A can not avoid the disturbance about B caused by the measurement A, when we consider two noncommuting observables A and B.

- **Entropic uncertainty relations**

In quantum informtion theory, in 1983 the entropic uncertainty relation was firstly introduced by Deustch [5] and improved by Uffink[6] in 1988. When probability distribution is defined as $X = \{p_1, p_2...p_n\}$ and $Y = \{q_1, q_2...q_n\}$ and $H(X)$ is Shannon entropy described as $H(X) = -\Sigma_i p_i \log p_i$, the entropic uncertainty relation is

$$H(X) + H(Y) \geq -2\log c, \quad \text{where } c = |\max_{i,j}\langle x_i|y_j\rangle| \tag{4}$$

where $\{|x_i\rangle\}$ and $\{|y_j\rangle\}$ are the corresponding complete sets of normalized eigenvectors with respect to operators X and Y, and then p_i, q_j is defined as $|\langle x_i|\psi\rangle|^2$ and $|\langle y_i|\psi\rangle|^2$. In the relation (4), the lower bound $-2\log c$ is independet of the initial state $|\psi\rangle$, but contrastively the relations (2)$\sim$(3) in the previous section is dependent. That means the lower bound, $|\langle\psi|[A,B]|\psi\rangle|$, is varied with an initail state, so in a specific state it is diminishing.

- **Entropic uncertainty relation for successive measurements**

In order to deal with unavoidable disturbance, we have to consider successive measurements, liks as the relation (3). In successive measurements, we obtain results about X and Y, in regular sequence. That means we observe the result of measurment X and the outcome will be measured for observable Y, so we get the results of X and Y finally. This situation is described in Fig 1.

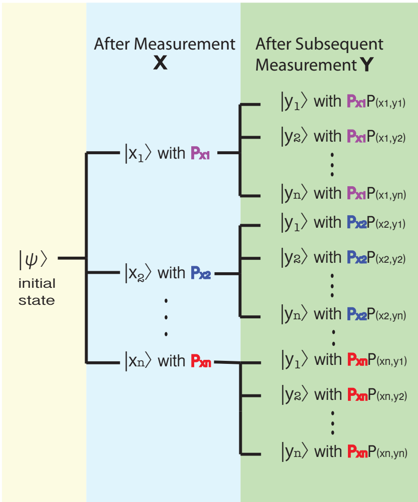

Figure 1: Probability distribution in successive measurements X and Y. Eigenvalue set of X,Y is $\{x_i\}$,$\{y_i\}$ and each eigenvector that corresponds to the eigenvalues x_i, y_i is $|x_i\rangle$ and $|y_i\rangle$. In this assumption, P_{xi} is defined as $|\langle\psi|x_i\rangle|^2$ and $P_{(xi,yj)}$ means $|\langle x_i|y_j\rangle|^2$

According to the Fig 1, a probability to obtain an eigenvalue x_i of eigenvector $|x_i\rangle$ after the measurement X is p_{xi}. Thus, the successive outcomes x_i and y_j are obtained with the product of each probabilty, $P_{xi}P_{(xi,yj)} = |\langle\psi|x_i\rangle|^2|\langle x_i|y_j\rangle|^2$. The probability distribution of whole possible outcomes is depicted in Fig 1.

Thus, the entropy of the probability distribution for the successive measuremnts is

$$H(X,Y) = -\sum_{i,j} p_{xi}p_{(xi,yj)} \log p_{xi}p_{(xi,yj)} \tag{5}$$

$$= H(X) - \sum_{i,j} p_{xi}p_{(xi,yj)} \log p_{(xi,yj)}. \tag{6}$$

$H(X,Y)$ quantifies naturally occuring uncertainty when a state is measured for observables X and Y in succession. In the situation, from the statement (iii) and the noise-disturbance relations, it can be easily conjectured that it is also limited by a lower bound, likewise with the entropic uncertainty relation. Hence, by simple calculation, the relation is

$$H(X) - \sum_{i,j} p_{xi}p_{(xi,yj)} \log p_{(xi,yj)} \geq -2\log c, \tag{7}$$

where $c = \max_{i,j}|\langle x_i|y_j\rangle|$.

References

[1] Heisenberg, W., *Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik*, Zeitschrift fur Physik, **43**, (3-4) 1927, 172-198.

[2] Robertson, H. P. The uncertainty principle. Phys. Rev. 34, 163164 (1929)

[3] Jacqueline Erhart, Stephan Spona, Georg Sulyok, Gerald Badurek, Masanao Ozawa, Yuji Hasegawa , *Experimental demonstration of a universally valid error-disturbance uncertainty relation in spin measurements*, Nature Physics, NPHY2194, 15 January, 2012

[4] Paul B., Teiko H., Pekka L. *Hesienber's uncertaintry principle*, Physics Reports, 452, 155 176 (2007)

[5] D. Deutsch *Uncertainty in quantum measurements* Phy. Rev. Lett. 50, 631 (1983)

[6] Hans Maasen, J. B. M. Uffink. *Generalized entropic uncertainty relations* Phy. Rev. Lett. 60, 1103-1106 (1988)

[7] Ozawa, M. *Uncertainty relations for noise and disturbance in generalized quantum measurements.* Ann. Phys. 311, 350416 (2004).

[8] Ozawa, M. *Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement*. Phy. Rev. A, 67, 042105 (2003)

[9] C. E. Shannon,*A mathmatical theory of communication* Bell system technical journal 27, 379 (1948).