Quantifying non-classical correlations via moment matrix positivity

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Introduction and our main result: Probabilities of measurement outcomes within the quantum framework are fundamentally different from those arising in the classical statistical scenario. This has invoked a wide range of debates on the foundational conflicts about the quantum-classical worldviews of nature [1, 2]. Pioneering investigations by Bell [3], Kochen-Specker [4], Leggett-Garg [5] tied the puzzling quantum features with various no-go theorems. The common aspect underlying the proofs of these no-go theorems points towards the non-existence of a joint probability distribution for the outcomes of all possible measurements performed on a quantum system [2–9].

On an entirely different perspective, *classical moment problem* [10, 11] addresses the issue of determining a probability distribution given a sequence of statistical moments. Essentially, classical moment problem brings forth that a given sequence of real numbers qualify to be moment sequence of a legitimate probability distribution if and only if the associated moment matrix is positive. In other words, *existence of a valid joint probability distribution* consistent with a given sequence of *moments* gets linked with moment matrix positivity.

In this work, we investigate positivity of 8×8 moment matrix to verify the existence of a valid joint probability distribution of three dichotomic observables in the quantum scenario. This results in an interesting identification: *positivity of the moment matrix implies*

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that the associated joint probabilities assume convex product form. In other words, hidden variable structure for the joint probabilities emerges naturally – bringing forth a necessary and sufficient condition for non-classicality of correlations via moment matrix positivity criterion. We propose a quantification of non-classical correlations based on the tracenorm of the *normalized* moment matrix. Specific examples of temporal correlations of a single qubit observable at three different times, when the system is evolving (a) under a coherent unitary dynamics and (b) an open system evolution resulting in amplitude damping are examined – with implications on Leggett-Garg macrorealism in terms of moment matrix quantification. We also explore the Bell non-locality, in conjunction with contextuality, in terms of the moment matrix associated with the statistical correlations of three dichotomic observables of a spatially separated two qubit system.

Outline of our approach: We consider three dichotomic variables X_1, X_2, X_3. A sequence of eight moments $\{1, \langle X_1\rangle, \langle X_2\rangle, \langle X_3\rangle, \langle X_1X_2\rangle, \langle X_2X_3\rangle, \langle X_1X_3\rangle, \langle X_1X_2X_3\rangle\}$ faithfully encode the details of the joint probability distribution $P(x_1, x_2, x_3),\ x_i = \pm 1$ [12]. We restrict to the specific case where $\langle X_i\rangle = 0,\ \langle X_1X_2X_3\rangle = 0$. A sequence of eight real numbers $\{1, 0, 0, 0, a, b, c, 0\}$, admit a valid classical probability districution $P(x_1, x_2, x_3)$, provided the 8×8 moment matrix, constructed using this set of numbers, is positive. The eigenvalues of the moment matrix are identified to be $\lambda_{1,2} = 1 + a - b - c, \lambda_{3,4} = 1 - a + b - c, \lambda_{5,6} = 1 - a - b + c$ and $\lambda_{7,8} = 1 + a + b + c$. We find that by embedding the sequence of numbers in the form of a three qubit density matrix (following the mapping $X_1 = \langle\vec{\sigma}\cdot\hat{n}_1 \otimes I \otimes I\rangle,\ X_2 \to I \otimes \vec{\sigma}\cdot\hat{n}_2 \otimes I,\ X_3 \to I \otimes \vec{\sigma}\cdot\hat{n}_3 \otimes I)$, positivity of the density matrix matches exactly with that of the moment matrix. In other words, we find an interesting connection that the three qubit density matrix is not a physical one – but is a *pseudo* density matrix when the corresponding sequence does not admit a valid three variable joint probability distribution [13]. However, the nature of joint probability distribution – when the corresponding sequence happens to be a physically valid set – is still left open. Significantly, we find that there is yet another faithful connection between the positivity of the moment matrix and that of a *partially transposed two qubit density matrix.* Positivity of partial transpose criterion [14] comes to help now – and it ascertains that *admissiblity of a joint probability distribution with the given sequence (moment matrix positivity) is ensured if and only if the associated two qubit density matrix is separable.* This in turn leads to our main identification that the given set of *moments* should necessarily allow a convex product decomposition of the joint probabilities, so as to be declared as a physically valid sequence of moments. A sequence obtained from *all possible* correlation measurement outcomes in the quantum scenario results, in general, in a pseudo moment matrix. A valid sequence of correlations – resulting within the quantum framework – necessarily implies hidden variable structure for the grand joint probabilities. A criterion based on the tracenorm of a normalized moment matrix (which agrees identically with the tracenorm of the pseudo three qubit density matrix associated with it) is useful to discern non-classsical correlations.

Conclusion: We have shown that the question of *existence of a grand joint probabilities* underlying all possible measurements of three dichotomic observables finds an elegant connection with the positivity of the associated moment matrix. Employing well-established results from quantum information theory we prove that positivity of the moment matrix necessarily implies a hidden variable structure for the joint probabilities. We have proposed a criterion based on the tracenorm of the moment matrix to quantify non-classical correlations. Examples of spatial and temporal correlations arising in the quantum scenario illustrate our results.

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[13] In a very recent work (J. Fitzsimons, J. Jones, and V. Vedral, arXiv:1302.2731) it has been highlighted that fitting the temporal correlations of dichotomic observables in the form of a multiqubit density matrix results in a psuedo (non-positive) density matrix. The departure of the associated pseudo density matrix from being physical has been employed to identify intrinsic non-classicality of temporal correlations.

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