When do pieces determine the whole? Extreme marginals of a completely positive map

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Abstract

This contribution is based on Ref [6] where we consider completely positive maps defined on tensor products of von Neumann algebras and taking values in the algebra of bounded operators on a Hilbert space and particularly certain convex subsets of the set of such maps. We show that when one of the marginal maps of such a map is an extreme point, then the marginals uniquely determine the map. We will further prove that when both of the marginals are extreme, then the whole map is extreme. We show that this general result is the common source of several well-known results dealing with, e.g., jointly measurable observables. We also obtain new insight especially in the realm of quantum instruments and their marginal observables and channels.

1 Introduction

Quantum devices and other objects in quantum theory often consist of several pieces operating on subsystems or on different layers of description or precision. Typical example of this are quantum instruments that describe measurement statistics and state changes conditioned by the outcomes. This means that a quantum instrument can be seen to consist of (at least) two parts; a map that associates a quantum state to a probability distribution (an observable) and state change (channel). Another example is states on multipartite systems that can be associated to the reduced states on the subsystems. Both of these examples illustrate the fact that pieces do not typically determine the whole; the observable and channel associated with a common instrument do not usually have a unique joint instrument. Likewise, states on subsystems generally have infinitely many possible joint states.

However, in some special cases pieces can be combined into a whole in a unique way. In [6], one such condition is found to be extremality of a piece and several results are shown to fall under this theme. By an extreme object we mean an extreme point in the convex set of all similar objects. (This total set can be e.g. the set of states, observables, instruments or channels.) We consider objects consisting of two pieces. Our main result, states, roughly, that *when one of the pieces is an extreme object, then pieces uniquely determine the whole.* Moreover, *when both of the pieces are extreme objects, then the whole is an extreme object.*

We recall the following well-known results that exemplify the previously sketched ideas.

(a) *Joint state with a pure marginal state*: Suppose that ϱ is a state of a composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$. If one of the reduced states $\mathrm{tr}_{\mathcal{H}_2}[\varrho] \equiv \varrho_1$ or $\mathrm{tr}_{\mathcal{H}_1}[\varrho] \equiv \varrho_2$ is pure, then $\varrho = \varrho_1 \otimes \varrho_2$. If both ϱ_1 and ϱ_2 are pure, then also ϱ is pure.

(b) *Joint observable with a sharp marginal observable*: Suppose that M and N are jointly measurable observables (POVMs). If M or N is sharp (i.e. projection valued measure), then their joint observable J is unique and it is determined by the condition $\mathsf{J}(X \times Y) = \mathsf{M}(X)\mathsf{N}(Y)$ for all outcome sets X, Y. (See e.g. [8] for a proof of this fact.)

(c) *Instruments related to a sharp observable*: Suppose that an observable M is sharp and Γ is an instrument such that $\Gamma(X, I) = \mathsf{M}(X)$ for all outcome sets X. Then $\Gamma(X, A) = \mathsf{M}(X)\mathcal{E}(A)$, where $\mathcal{E} = \Gamma(\Omega, \cdot)$ is the total state transformation and Ω is the total set [9]. Hence the instrument Γ is completely determined by its total state transformation $\mathcal{E}$.

(d) *Variant of 'No Cloning Theorem'*: Suppose $\mathcal{F} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H} \otimes \mathcal{H})$ is a quantum channel such that $\mathrm{tr}_1[\mathcal{F}(\varrho)] = \varrho$ for every state ϱ. Then $\mathrm{tr}_2[\mathcal{F}(\varrho)] \equiv \sigma$ for some fixed state σ, hence the attempted copy $\mathrm{tr}_2[\mathcal{F}(\varrho)]$ contains no information on the input state ϱ.

Theorem 1 proven in [6] contains all these statements as corollaries and identifies the common source behind the uniqueness claims as being extremality of a marginal map. Also some completely new results are obtained. Our main theorem implies the following:

(e) Suppose that M and N are jointly measurable observables (POVMs). If M or N is extreme, then their joint observable is unique. If both M and N are extreme, then their unique joint observable is extreme.

(f) Suppose that an observable M and a channel $\mathcal{E}$ are parts of a single instrument Γ, i.e., $\Gamma(X, I) = \mathsf{M}(X)$ for all outcome sets $X \subseteq \Omega$ and $\Gamma(\Omega, \cdot) = \mathcal{E}$ for the total set Ω. If M or $\mathcal{E}$ is extreme, then the instrument Γ is unique. If M and $\mathcal{E}$ are both extreme, then Γ is extreme.

The last two consequences of the main theorem are new, since, e.g., there are extreme observables that are not sharp [7]. It is often noted that extreme quantum apparati exhibit less noise than others because they contain no arbitrariness caused by randomization between different preparation or measurement strategies. Still, it is unclear whether extremality has a concrete operational meaning in quantum theory. Our result, Theorem 1 below however, gives new essence to extremality and could be used to find what extremality of a quantum device truly means.

2 Results

We will not go into mathematical details in this exposition, but let us make some definitions (more rigorous formulation can be found in [6]). We study completely positive (CP) maps Φ defined on a von Neumann algebra $\mathcal{A}$ and taking values in the set $\mathcal{L}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. Complete positivity means, roughly, that we can dilate the map Φ into a *positive* map $\Phi \otimes \mathrm{id}_n$ (i.e., $\Phi \otimes \mathrm{id}_n$ maps positive elements into positive elements) defined on a $n \times n$-matrix algebra over $\mathcal{A}$ for any value $n = 1, 2, \ldots$ and taking values in $\mathcal{L}(\mathcal{H} \otimes \mathbb{C}^n)$. In physical situations this means that when we couple the system under study with another system of arbitrary size, our map (observable, channel, instrument,...) can be trivially dilated into a positive map on the whole without disturbing the other system.

We further impose the condition $\Phi(1_{\mathcal{A}}) = P$ with a fixed positive operator $P \in \mathcal{L}(\mathcal{H})$ on our CP maps. This set of P-normalized CP maps is denoted by $\mathbf{CP}_P(\mathcal{A}; \mathcal{H})$. This is a convex set: when $\Phi_1, \Phi_2 \in \mathbf{CP}_P(\mathcal{A}; \mathcal{H})$ and $0 \leq t \leq 1$ we can form a convex combination $t\Phi_1 + (1-t)\Phi_2 \in \mathbf{CP}_P(\mathcal{A}; \mathcal{H})$. As usual, extreme points of $\mathbf{CP}_P(\mathcal{A}; \mathcal{H})$ are those Φ that cannot be expressed as a convex combination of non-equal elements with weights $t \in (0, 1)$. Extreme points have been characterized, e.g., in [1], [6], [10] and [11].

When also $\mathcal{B}$ is a von Neumann algebra, we can define the tensor product $\mathcal{A} \otimes \mathcal{B}$ which is a von Neumann algebra, too. Suppose that $\Psi \in \mathbf{CP}_P(\mathcal{A} \otimes \mathcal{B}; \mathcal{H})$. With inputs of the form $a \otimes 1_{\mathcal{B}}$, $a \in \mathcal{A}$, we obtain a map $\Psi_{(1)} \in \mathbf{CP}_P(\mathcal{A}; \mathcal{H})$ and imputs $1_{\mathcal{A}} \otimes b$, $b \in \mathcal{B}$, induce a map $\Psi_{(2)} \in \mathbf{CP}_P(\mathcal{B}; \mathcal{H})$. We call the maps $\Psi_{(1)}$ and $\Psi_{(2)}$ as *marginals* of Ψ. Moreover, if $\Phi_1 \in \mathbf{CP}_P(\mathcal{A}; \mathcal{H})$ and $\Phi_2 \in \mathbf{CP}_P(\mathcal{B}; \mathcal{H})$ are marginals of some map $\Psi \in \mathbf{CP}_P(\mathcal{A} \otimes \mathcal{B}; \mathcal{H})$, i.e., $\Phi_1 = \Psi_{(1)}$ and $\Phi_2 = \Psi_{(2)}$, we say that Φ_1 and Φ_2 are *compatible*. Moreover, in this case the map Ψ is called as the *joint map* for Φ_1 and Φ_2.

The quantum instruments, e.g., can be viewed as (normal) CP maps defined on the tensor product of a (commutative) function algebra and $\mathcal{L}(\mathcal{K})$ with some Hilbert space $\mathcal{K}$. The first marginal w.r.t. the commutative algebra (meaning we ignore the state changes) gives the observable associated with the instrument whereas the second marginal corresponds to the total state change induced by the channel obtained by coarse graining over the outcomes of the measurement. An observable and a channel are compatible if they can be obtained in this way from a joint instrument. Observables M_1 and M_2 with outcome spaces Ω_1 and, respectively, Ω_2 are compatible or *jointly measurable* if there exists an observable with outcomes in $\Omega_1 \times \Omega_2$ such that coarse graining over Ω_2 gives M_1 and coarse graining over Ω_1 produces M_2. The marginals of a state on a multipartite system are simply the reduced states on the subsystems; note that states are automatically compatible. The definition of compatibility of CP maps briefly sketched earlier encompasses all the above definitions of compatibility of quantum apparati.

Our main result is the following:

Theorem 1 *Suppose that $\Phi_1 \in \mathbf{CP}_P(\mathcal{A};\mathcal{H})$ and $\Phi_2 \in \mathbf{CP}_P(\mathcal{B};\mathcal{H})$ are compatible.*

(a) *If Φ_1 is extreme in $\mathbf{CP}_P(\mathcal{A};\mathcal{H})$ or Φ_2 is extreme in $\mathbf{CP}_P(\mathcal{B};\mathcal{H})$, then they have a unique joint map.*

(b) *If Φ_1 is extreme in $\mathbf{CP}_P(\mathcal{A};\mathcal{H})$ and Φ_2 is extreme in $\mathbf{CP}_P(\mathcal{B};\mathcal{H})$, then their unique joint map is extreme in $\mathbf{CP}_P(\mathcal{A}\otimes\mathcal{B};\mathcal{H})$.*

(c) *If Φ_1 or Φ_2 is a *-representation, then Φ_1 and Φ_2 commute and the unique joint map $\Psi \in \mathbf{CP}_P(\mathcal{A}\otimes\mathcal{B};\mathcal{H})$ is of the form*

$$\Psi(a \otimes b) = \Phi_1(a)\Phi_2(b), \qquad a \in \mathcal{A}, \quad b \in \mathcal{B}. \tag{1}$$

Since the sets of quantum states, observables, channels, and instruments can be seen as (suitable subsets of) $\mathbf{CP}_P(\mathcal{A};\mathcal{H})$ with appropriate choices of the algebra and the Hilbert space, Theorem 1 has all the results listed in the Introduction, especially the novel findings (e) and (f), as its corollaries.

Theorem 1 tells that if one of the pieces is extreme, then the whole is perfectly determined by the pieces, but, typically, we do not know the exact structure of the whole. There are, however, situations, where we can say something about the structure of the joint map. Let us study channels $\mathcal{E} : \mathcal{S}(\mathcal{H}_1\otimes\mathcal{H}_2) \to \mathcal{S}(\mathcal{K}_1\otimes\mathcal{K}_2)$ between bipartite systems where $\mathcal{S}(\mathcal{H})$ is the set of states on Hilbert space $\mathcal{H}$. These maps are CP and normal; technically, their duals $\mathcal{E}^* : \mathcal{L}(\mathcal{K}_1\otimes\mathcal{K}_2) \to \mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ are CP, unital, and normal. If $\mathcal{E}$ is such that if with imputs of the form $\rho_1\otimes\rho_2$, $\rho_r \in \mathcal{S}(\mathcal{H}_r)$, and after applying $\mathcal{E}$ tracing out the system $\mathcal{K}_2$ (resp. $\mathcal{K}_1$) we obtain a channel $\mathcal{E}^1 : \mathcal{S}(\mathcal{H}_1) \to \mathcal{S}(\mathcal{K}_1)$ (resp. $\mathcal{E}^2 : \mathcal{S}(\mathcal{H}_2) \to \mathcal{S}(\mathcal{K}_2)$), we say, adopting the terminology introduced in [2] and [5], that $\mathcal{E}$ is *causal*. Channels of the form $\mathcal{E}(\rho_1\otimes\rho_2) = \mathcal{E}^1(\rho_1)\otimes\mathcal{E}^2(\rho_2)$, $\rho_r \in \mathcal{S}(\mathcal{H}_r)$, where $\mathcal{E}^r : \mathcal{S}(\mathcal{H}_r) \to \mathcal{S}(\mathcal{K}_r)$ are channels are called *local*. Local channels are clearly causal, but a causal channel needs not to be even in the convex hull of local channels (see examples in [2] and [3]). However, Theorem 1 tells that if one of the subchannels of a causal channel is extreme, then the whole is local.

Theorem 2 *Suppose that a causal channel $\mathcal{E}$ has the channels $\mathcal{E}^1$ and $\mathcal{E}^2$ as its subchannels in the above defined way. If either of $\mathcal{E}^1$ or $\mathcal{E}^2$ is extreme, then $\mathcal{E}$ is local.*

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