Measuring Time-Energy Resources for Quantum Processes

Chi-Hang Fred Fung[1,*] and H. F. Chau[1]

[1] *Department of Physics and Center of Theoretical and Computational Physics, University of Hong Kong, Pokfulam Road, Hong Kong*

The speed of any quantum process is limited by quantum mechanics via time-energy uncertainty relations and they imply that time and energy are tradeoff against each other. As such, in this paper, we propose to measure the time-energy as a single unit for quantum channels. We consider a time-energy measure for quantum channels and compute lower and upper bounds of it using the channel Kraus operators. For a special class of channels (which includes the depolarizing channel), we can obtain the exact value of the time-energy measure. Our result can be used to compare the time-energy resources of similar quantum processes. In particular, we show that erasing quantum information requires $\sqrt{(n+1)/n}$ times more time-energy resource than erasing classical information, where n is the system dimension.

Quantum processes in nature and quantum computation processes designed by human all require time and energy to evolve. The evoluation speed of a physical device is governed by physical laws and is limited by the energy of the device. Under the constraints of quantum mechanics, time-energy uncertainty relations (TEURs) set limits on system evolutions [1]. The investigation of TEURs has a long history. Mandelstam and Tamm [2] proved the first major result of a TEUR. This was followed by subsequent work on isolated systems [3–10] and composite systems with entanglement [11–13]. Recently, TEURs for general quantum processes have also been proved [14, 15]. The general form of TEURs is an inequality that sets a lower limit on the product of the system energy (or a function of the energies) and the time it takes to evolve an initial state to a final state (e.g., an orthogonal state). One implication of the form of TEURs is that time and energy are tradeoff against each other. Thus, we propose to regard time-energy as a single property of a quantum process. The intuition is that the more computation or work a quantum process performs, the more time-energy it requires. And it is up to the system designer (or nature) to perform it with more time but less energy, or vice versa. Thus, our goal in this paper is to investigate the time-energy requirements of quantum processes by using a time-energy measure. Research in this direction has been carried out before. Chau [16] proposed a time-energy measure for unitary transformations that is based on a TEUR proved earlier [10]. In this paper, we extend this measure to quantum processes. The TEUR due to Chau [10] is tight in the sense that it can be saturated by some states

* chffung@hku.hk

and Hamiltonians, and thus it serves to motivate a good definition for a time-energy measure. To see this, let's start with this TEUR. Given a time-independent Hamiltonian H of a system, the time t needed to to evolve a state $|\Phi\rangle$ under the action of H to a state whose fidelity [17] is less than or equal to ϵ satisfies the TEUR

$$t \geq \frac{(1-\sqrt{\epsilon})\hbar}{A\sum_j |\alpha_j|^2 |E_j|} \tag{1}$$

where E_j's are the eigenvalues of H with the corresponding normalized energy eigenvectors $|E_j\rangle$'s, $|\Phi\rangle = \sum_j \alpha_j |E_j\rangle$, and $A \approx 0.725$ is a universal constant. Essentially, after time t, the state transforms unitarily according to $U = e^{-iHt/\hbar}$. The same U could be implemented with either a high energy H run for a shorter time or a low energy H run for a longer time. Based on Eq. (1), a weighted sum of $|tE_j|$'s serves as an indicator of the time-energy resource needed to perform U, and as such the following time-energy measure on unitary matrices was proposed by Chau [16]:

$$\|U\|_{\vec{\mu}} = \sum_{j=1}^{r} \mu_j |\theta_j|^{\downarrow}$$

where U has eigenvalues $\exp(-iE_j t/\hbar) \equiv \exp(\theta_j)$, and $|\theta_j|^{\downarrow}$ denotes $|\theta_j|$ ordered non-increasingly $|\theta_1|^{\downarrow} \geq |\theta_2|^{\downarrow} \geq \cdots \geq |\theta_r|^{\downarrow}$. Also, $\vec{\mu} = [\mu_1, \mu_2, \ldots, \mu_r] \neq \vec{0}$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0$. Note that $\|U\|_{\vec{\mu}}$ satisfies the multiplicative triangle inequality $\|UV\|_{\vec{\mu}} \leq \|U\|_{\vec{\mu}} + \|V\|_{\vec{\mu}}$ [16]. In essence, a large value of $\|U\|_{\vec{\mu}}$ suggests that a long time may be needed to run a Hamiltonian that implements U for a fixed energy, and vice versa.

In this paper, we are interested in an analogous measure for quantum channels which include unitary transformations as special cases. We are given a quantum channel $\mathcal{F}(\rho)$ acting on system A that maps $n \times n$ density matrix ρ to another one with the same dimension. There exist unitary extensions U_{BA} in a larger Hilbert space with an ancillary system B such that $\mathcal{F}(\rho) = \mathrm{Tr}_B[U_{BA}(|0\rangle_B\langle 0| \otimes \rho_A)U_{BA}^\dagger]$. Each U_{BA} could have a different time-energy spectrum and we want to select the one requiring the least resource for $\mathcal{F}$. We extend the resource indicator for U to quantum channel $\mathcal{F}$ by defining

$$\|\mathcal{F}\|_{\vec{\mu}} \equiv \min_U \|U\|_{\vec{\mu}}$$
$$\text{s.t.}\, \mathcal{F}(\rho) = \mathrm{Tr}_B[U_{BA}(|0\rangle_B\langle 0| \otimes \rho_A)U_{BA}^\dagger]\, \forall \rho.$$

This gives a U that consumes the least time-energy resource. Thus, $\|\mathcal{F}\|_{\vec{\mu}}$ is an indicator of the resource needed to perform $\mathcal{F}$.

There are some interesting consequences by using this time-energy measure. In particular, we can compare the time-energy resources needed to erase quantum information and classical information. We show that $\sqrt{(n+1)/n}$ times more resource is required in the quantum setting than in the classical setting. Also, we study the time-energy scaling of consecutive runs of the depolarizing channel. It turns out that the amount of time-energy resource needed for k runs of the depolarizing channel scales as $\sqrt{k}$ when the noise is small.

In this work, we focus on two special cases of the time-energy measure:

- Sum time-energy: $\|U\|_{\text{sum}} \equiv \sum_{j=1}^{r} |\theta_j| \ .$
- Max time-energy: $\|U\|_{\text{max}} \equiv \max_{1\le j\le r} |\theta_j| = |\theta_1|^{\downarrow} \ .$

Note that the subscript "sum" is short for $\vec{\mu} = [1, 1, \ldots, 1]$ and "max" for $\vec{\mu} = [1, 0, \ldots, 0]$.

For these two cases, we derive lower and upper bounds on the time-energy resource measure $\|\mathcal{F}\|_{\text{max}}$ and $\|\mathcal{F}\|_{\text{sum}}$ for any quantum channel $\mathcal{F}$ given its Kraus operators. The derivation is based on analyzing a few intermediate optimization problems. It turns out that the lower and upper bounds are all dependent on the eigenvalues of some Kraus operator of $\mathcal{F}$. Specifically, we prove that

$$\|\mathcal{F}\|_{\text{max}} \ge \min_{\mathbf{v}:\, \|\mathbf{v}\|\le 1} \max_{1\le i\le n} \cos^{-1}\Big[\text{Re}(\lambda_i(\sum_{j=1}^{d} v_j F_j))\Big] \tag{2}$$

$$\|\mathcal{F}\|_{\text{max}} \le \min_{\mathbf{v}:\, \|\mathbf{v}\|\le 1} \sum_{i=1}^{n} \cos^{-1}\Big[\text{Re}(\lambda_i(\sum_{j=1}^{d} v_j F_j))\Big] \tag{3}$$

$$\|\mathcal{F}\|_{\text{sum}} \ge \min_{\mathbf{v}:\, \|\mathbf{v}\|=1} \max_{1\le i\le n} 2\cos^{-1} |\lambda_i(\sum_{j=1}^{d} v_j F_j)| \tag{4}$$

$$\|\mathcal{F}\|_{\text{sum}} \le \min_{\mathbf{v}:\, \|\mathbf{v}\|\le 1} \sum_{i=1}^{n} 2\cos^{-1}\Big[\text{Re}(\lambda_i(\sum_{j=1}^{d} v_j F_j))\Big] \tag{5}$$

where $\|\mathbf{v}\| = \sqrt{\sum_{j=1}^{d} |v_j|^2}$, $F_j \in \mathbb{C}^{n\times n}, j = 1, \ldots, d$ are the Kraus operators of $\mathcal{F}$, and $\lambda_i(\cdot)$ denotes the ith eigenvalue of its argument. Note that $\mathcal{F}(\rho) = \sum_{j=1}^{d} F_j \rho F_j^{\dagger}$.

For a class of channels (which includes the depolarizing channel), we obtain the exact value for $\|\mathcal{F}\|_{\text{max}}$. In particular, when $\mathcal{F}$ is a depolarizing channel with probability q that the input state is unchanged, its time-energy requirement is $\|\mathcal{F}\|_{\text{max}} = \cos^{-1}\sqrt{q + (1-q)/n^2}$.

We summarize the approach used to establish the lower and upper bounds. We cast the original problem of finding the most time-energy efficient U that implements $\mathcal{F}$ as the problem of finding a U that transforms some given initial vectors to some given final vectors. The time-energy of this U is lower bounded by that of any U' that transforms a subset of the vectors. This is essence of how we obtain a lower bound for $\|\mathcal{F}\|_{\text{max}}$ and $\|\mathcal{F}\|_{\text{sum}}$, by searching for U' that transforms only one vector. To derive the upper bound, we construct a sequence of single-vector transformations such that their product gives the original U. Since $\|U\|_{\vec{\mu}}$ satisfies the multiplicative triangle inequality $\|UV\|_{\vec{\mu}} \le \|U\|_{\vec{\mu}} + \|V\|_{\vec{\mu}}$ [16], we obtain the upper bound. Thus, both the derivations of the lower and upper bounds rely on the solution to the single-vector-transformation problem.

We make a few other remarks. A related concept about erasure and energy is the Landauer's principle [18] which puts lower limits on the energy dissipated to the environment in erasing (qu)bits. There is a difference between the erasure considered here and the erasure of the Landauer's principle. For future investigation, it is instructive to obtain the time-energy for var-

ious quantum processes such as some standard gates or algorithms, to consider this time-energy measure in the thermodynamic setting, and to explore deeper operational meaning about this measure.

[1] S. Lloyd, Nature **406**, 1047 (2000).

[2] L. Mandelstam and I. Tamm, J. Phys. USSR **9**, 249 (1945).

[3] K. Bhattacharyya, J. Phys. A **16**, 2993 (1983).

[4] J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65**, 1697 (1990).

[5] A. Uhlmann, Phys. Lett. A **161**, 329 (1992).

[6] L. Vaidman, Am. J. Phys. **60**, 182 (1992).

[7] P. Pfeifer, Phys. Rev. Lett. **70**, 3365 (1993).

[8] N. Margolus and L. B. Levitin, in *Proceedings of Fourth Workshop on Physics and Computation*, edited by T. Toffoli, M. Biafore, and J. Leão (New England Complex Systems Institute, Boston, 1996) pp. 208–211.

[9] N. Margolus and L. B. Levitin, Physica D **120**, 188 (1998).

[10] H. F. Chau, Phys. Rev. A **81**, 062133 (2010).

[11] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. A **67**, 052109 (2003).

[12] V. Giovannetti, S. Lloyd, and L. Maccone, Europhys. Lett. **62**, 615 (2003).

[13] C. Zander, A. R. Plastino, A. Plastino, and M. Casas, J. Phys. A **40**, 2861 (2007).

[14] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Phys. Rev. Lett. **110**, 050402 (2013).

[15] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Phys. Rev. Lett. **110**, 050403 (2013).

[16] H. F. Chau, Quant. Inf. Compu. **11**, 0721 (2011).

[17] We adopt the fidelity definition $F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right)^2$ for two quantum states ρ and σ.

[18] R. Landauer, IBM J. Res. Dev. **5**, 183 (1961).