Optimal fidelity for qubit channels

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1 Introduction

One of the major goals in information theory, classical or quantum, is to make optimal use of the communication resources. The capacity of a channel thus reflects the best rate of transmission of bits/qubits that can be achieved, with negligible error, for many uses of the channel [1, 2]. While capacities are well known in the classical case, finding capacities of quantum channels is believed to be hard [2]. Alternatively, one can characterize quantum channels through the notion of fidelity [11], which measures, on an average, how close the output state is to the input state.

The concepts of quantum channel capacity or fidelity aim to capture, in a physically meaninful way, the channel's inherent ability to transmit quantum information. Nevertheless, their usefulness is limited because in most cases they are difficult to compute. On the other hand, quantum channels can also be used to share entanglement [3] between two remote observers. This is particularly important in the context of quantum information theory, where shared entanglement is a critical resource of quantum information tasks, especially quantum teleportation [4]. Thus the problem of how well a quantum channel can trasnmit a quantum state can also be posed as how well a quantum channel can serve as a resource for quantum information processing tasks (e.g. teleportation). It is in this spirit we ask the following question: Given a quantum channel, what is the maximum fidelity (singlet fraction) of an entangled state that we can possibly prepare by means of trace preserving local operations (TP LOCC), where the maximum is taken over all such preparations? Noting that there is a direct relation between the teleportation fidelity and singlet fraction of any bipartite density matrix [11], the answer to the above question immediately gives us the optimal teleportation fidelity for a quantum channel. While the answer is known for channels such as depolarizing [11] and amplitude damping [12], it remains open for a generic qubit channel. In this paper we give a complete answer to the above

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question for qubit channels and present an explicit protocol to achieve the optimal value. We also prove definite results on whether or not the optimal value is achieved by sending maximally entangled states in a general setting for sharing entanglement.

To define the optimal fidelity for a qubit channel we proceed as follows. First, Alice prepares a two qubit pure state $|\psi\rangle$ and sends part of it to Bob using the qubit channel Λ they share. This results in a mixed entangled state $\rho_{\psi,\Lambda}$, purity of which is measured by its fidelity of fully entangled fraction [7, 9, 11]

$$F\left(\rho_{\psi,\Lambda}\right) \quad = \quad \max_{|\Phi\rangle}\langle\Phi|\rho_{\psi,\Lambda}|\Phi\rangle, \tag{1}$$

where $|\Phi\rangle$ is a maximally entangled state. The notion of fidelity is of considerable interest in quantum information theory as it quantifies how close the state $\rho_{\psi,\Lambda}$ is to a maximally entangled state as well as how useful the state is for quantum teleportation [11] and entanglement distillation [11, 7, 9].

Next, Alice and Bob apply local trace preseving operations (TP-LOCC) on their respective qubits to increase the fidelity [14, 15, 16]. Thus the maximum achievable fidelity for a given $\rho_{\psi,\Lambda}$ is defined as[16]

$$F^*\left(\rho_{\psi,\Lambda}\right) \quad = \quad \max_{\text{TP-LOCC}} F\left(\rho_{\psi,\Lambda}\right) \tag{2}$$

Note that unlike fidelity, the maximum achievable fidelity is an entanglement monotone [16] and thus cannot be increased on an average by local operations and classical communication (LOCC) and moreover, it can be exactly computed by solving a convex semi-definite program for any given two-qubit density matrix [16]. The *optimal fidelity* of the qubit channel Λ is now obtained by maximizing F^* over all transmitted pure states $|\psi\rangle$:

$$F(\Lambda) \quad = \quad \max_{|\psi\rangle} F^*\left(\rho_{\psi,\Lambda}\right) \tag{3}$$

It is clear that for any transmitted pure state $|\psi\rangle$, the following inequalities hold:

$$F(\Lambda) \geq F^*\left(\rho_{\psi,\Lambda}\right) \geq F\left(\rho_{\psi,\Lambda}\right) \tag{4}$$

Now noting that the optimal teleportation fidelity $f(\sigma)$ of any two qubit density matrix σ is related to its fidelity via the following relation [11]:

$$f(\sigma) \quad = \quad \frac{2F(\sigma)+1}{3}. \tag{5}$$

we immediately get the *optimal teleportation fidelity* for the channel Λ

$$f(\Lambda) \quad = \quad \frac{2F(\Lambda)+1}{3}. \tag{6}$$

2 Statement of results

In this paper we consider only qubit channels, that is, quantum channels that allow transmission of qubits. Our first result concerns the maximum attainable fidelity for a qubit channel Λ.

Theorem 1. *Let $|\psi_0\rangle$ be the eigenvector of the largest eigenvalue of the density matrix $\rho_{\Phi^+,\hat{\Lambda}}$. Then the optimal fidelity of the channel Λ is given by*

$$F(\Lambda) = F^*\left(\rho_{\psi_0,\Lambda}\right) = F\left(\rho_{\psi_0,\Lambda}\right) = \lambda_{\max}\left(\rho_{\phi^+,\Lambda}\right), \tag{7}$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $\lambda_{\max}\left(\rho_{\Phi^+,\Lambda}\right)$ is the maximum eigenvalue of $\rho_{\Phi^+,\Lambda}$.

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Thus the optimal fidelity is given by the maximum eigenvalue of the density matrix $\rho_{\Phi^+,\Lambda}$ – the state obtained by sending half of the maximally entangled state $|\Phi^+\rangle$ through the channel. The above result also tells us explicitly how to attain this optimal value: one simply needs to transmit part of a *suitable* two qubit pure state $|\psi_0\rangle$ through the channel; the fidelity of the resulting state $\rho_{\psi_0,\Lambda}$ thus obtained is optimal and requires no further local post-processing. This *optimal* state $|\psi_0\rangle$ is given by the eigenvector corresponding to the maximum eigenvalue of $\rho_{\Phi^+,\hat{\Lambda}}$ – the state obtained by sending half of the state $|\Phi^+\rangle$ through the *dual* channel $\hat{\Lambda}$.

What can we say about $|\psi_0\rangle$? Evidences so far are mixed: $|\psi_0\rangle$ could be either maximally entangled (e.g., depolarizing channel [11]) or nonmaximally entangled (e.g., amplitude damping [12]), but the answer for a generic qubit channel is not known. The following result characterizes the channels for which $|\psi_0\rangle$ is maximally entangled and for which it is not.

Theorem 2. *The state $|\psi_0\rangle$ is maximally entangled if and only if the channel Λ is unital, where $|\psi_0\rangle$ is the eigenvector corresponding to the largest eigenvalue of $\rho_{\Phi^+,\hat{\Lambda}}$.*

Theorem 2 tells us that following the prescription of Theorem 1, the optimal fidelity cannot be obtained by sending part of a maximally entangled state if the channel is nonunital. We note that an instance of this result has been previously observed in the case of amplitude damping channel [12], which is known to be nonunital. On the other hand if we know that the optimal value is attained by sending part of a maximally entangled state, then we can certainly conclude that the channel is unital.

It is important to recognize that the above results, especially Theorem 1 only prescribes a method to attain the optimal fidelity. It does not, however, rule out the possibility that the optimal fidelity may still be attained by sending a maximally entangled followed by local post-processing. As it turns out this would not be the case as the next result shows.

Theorem 3. *For a nonunital channel Λ,*

$$F^*\left(\rho_{\Phi^+,\Lambda}\right) \quad < \quad F(\Lambda) \tag{8}$$

Thus for a nonunital channel the optimal fidelity is achieved only by sending nonmaximally entangled states through the channel.

References

[1] A. S. Holevo, V. Giovannetti, Quantum channels and their entropic characteristics, Rep. Prog. Phys. 75 046001 (2012)

[2] G. Smith and J. A. Smolin, Detecting Incapacity, Phys. Rev. Lett. 108, 230507 (2012).

[3] E. Schrodinger, Naturwissenschaften 23, 807 (1935); For a recent review see, R. Horodecki, P. Horodecki. M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. 81, 865 (2009).

[4] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. 70, 1895 (1993).

[5] M. Horodecki, P. W. Shor, and M. B. Ruskai, Rev. Math. Phys. 15, 629 (2003). DOI: 10.1142/S0129055X03001709

[6] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?, Phys. Rev. Lett. 80, 5239 (1998).

[7] C. H. Bennett, D. P. Divincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A 54, 3824 (1996).

[8] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu,, and A. Sanpera, Quantum privacy amplification and the security of quantum cryptography over noisy channels, Phys. Rev. Lett., 77, 2818 (1996).

[9] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of noisy entanglement and faithful teleportation via noisy channels, Phys. Rev. Lett. 76, 722 (1996).

[10] M. Horodecki, P. Horodecki, and R. Horodecki, Inseparable Two Spin- 1/2 Density Matrices Can Be Distilled to a Singlet Form, Phys. Rev. Lett. 78, 574 (1997)

[11] M. Horodecki, P. Horodecki, and R. Horodecki, General teleportation channel, singlet fraction, and quasidistillation, Phys. Rev. A 60, 1888 (1999).

[12] S. Bandyopadhyay and A. Ghosh, Optimal fidelity for a quantum channel may be attained by nonmaximally entangled states, Phys. Rev. A 86, 020304(R) (2012).

[13] F. Verstraete and H. Verschelde, Fidelity of mixed states of two qubits, Phys. Rev. A 66, 022307 (2002).

[14] P. Badziag, M. Horodecki, P. Horodecki, and R. Horodecki, Local environment can enhance fidelity of quantum teleportation, Phys. Rev. A 62, 012311 (2000).

[15] S. Bandyopadhyay, Origin of noisy states whose teleportation fidelity can be enhanced through dissipation, Phys. Rev. A 65, 022302 (2002).

[16] F. Verstraete and H. Verschelde, Optimal Teleportation with a Mixed State of Two Qubits, Phys. Rev. Lett. 90, 097901 (2003).

[17] F. Verstraete and H. Verschelde, arXiv:quant-ph/0202124v2.