Is 2-EPP good at low rates for phase damping channel ?

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Keywords: entanglement purification protocol, quantum error correcting codes, phase damping channel, reliability function, exponential bound

1 Introduction

Entanglement is known as a useful resource in quantum information processings. In particular, to share a maximally entangled state is important since there are many applications (e.g. [1, 2, 3]). In order to share a maximally entangled state via a noisy quantum channel, entanglement purification protcols(EPPs)[4] is essential. There are two classes of EPPs; 1-EPP which uses one-way classical communication and 2-EPP which uses two-way. Although 2-EPPs are superior to 1-EPPs in general settings, it seems that 1-EPPs are sufficient to purify entangled states degraded by a channel with error of one kind (e.g. phase-damping channel) since the upper limit of yield of 1-EPPs and that of 2-EPPs coincides. Recently, we showed that a 2-EPP is superior to a 1-EPP when fifteen entangled states are initially shared via a phase-damping channel. This result implies that a 2-EPP is superior to 1-EPPs when the number of initial shared entanglement is finite although any 2-EPP does not outperform the best 1-EPP in the infinite limit. Saying with the language of information theory, a 2-EPP is superior to 1-EPPs at much lower rate than the 'capacity' even if there is only one kind of error in the channel. In classical and quantum information theories, several examples are known that show superiority at low rates whereas capacities are not improved[8, 9, 10]. For example, in classical information theory, although it is well-known that channel capacities with and without feedback are identical [7], feedback is effective if the rate is lower than the capacity [8]. In quantum information theory, since classical capacity of attenuated quantum channel is attained by coherent states [9], it seems that utilizing squeezed states is meaningless. However, it was shown that squeezing is useful at low rates [10]. In either case, bounds on reliability function or error exponent are employed to show advantage at low rates.

In the present paper, we show property of exponential bounds on $1-F$ of EPPs, which correspond to bounds on error exponent in usual information transmission. As a result, it is clarified that an exponential bound of fidelity of a particular 2-EPP is higher than a bound of fidelity of a 1-EPP at least at low rates when the channel is assumed to be depolarizing or phase damping channels. The latter result clearly shows that a 2-EPP is superior to 1-EPPs and use of 2-EPP saves the number of initial shared entanglements even if there is only one kind of error.

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2 Basic notions

In this section, we explain the problem setting and describe channels, EPPs, and their evaluation.

In this study, we consider the following problem. Alice prepares n Bell states

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle), \tag{1}$$

and she sends half of each state to Bob over a noisy quantum channel. Here, $|0\rangle$ and $|1\rangle$ are orthonormal basis vectors in the qubit system. Then, Alice and Bob apply an EPP to them. If they keep their pairs, they share k entangled states.

2.1 Noisy quantum channels

First, we define a qubit channel based on Ref.[11]. Let σ be an input qubit state of the channel. Then the output state of the channel is described as

$$\begin{aligned}\mathcal{A}(\sigma) = P((0,0))\sigma + P((1,0))X\sigma X^\dagger \\ + P((0,1))Z\sigma Z^\dagger + P((1,1))(XZ)\sigma(XZ)^\dagger,\end{aligned} \tag{2}$$

where $u = (i,j) \in \{0,1\}^2$, $P(u)$ is probability which satisfies $0 \le P(u) \le 1$ and $\sum_u P(u) = 1$. As for sharing an entangled state, suppose the sender (Alice) transmits a half of the 2-qubit state $|\Phi^+\rangle$ to the receiver (Bob) over the qubit channel. Then the shared state is

$$\begin{aligned}\rho =& P((0,0))|\Phi^+\rangle\langle\Phi^+| + P((1,0))|\Psi^+\rangle\langle\Psi^+| \\ &+ P((0,1))|\Phi^-\rangle\langle\Phi^-| + P((1,1))|\Psi^-\rangle\langle\Psi^-|,\end{aligned} \tag{3}$$

where $|\Psi^\pm\rangle, |\Phi^\pm\rangle$ is the well-known four Bell states.

2.1.1 Depolarizing channel

A depolarizing channel corresponds to the case of $P((0,0)) = 1-p, P((0,1)) = P((1,0)) = P((1,1)) = p/3$ in Eq.(2).

2.1.2 Phase damping channel

A phase-damping channel corresponds to the case of $P((0,0)) = 1-p, P((1,0)) = p, P((0,1)) = P((1,0)) = 0$ in Eq.(2). As an example of the channel, there is an attenuation channel for a quasi-Bell state [12, 13] or an entangled coherent state (e.g. [14, 15]). Hence, a phase-damping channel is important for applications [16].

2.2 Entanglement purification protocols

The entanglement purification protocols(EPPs) are protocols to distil n pairs of mixed entangled states $\rho^{\otimes n}$ into k pairs of (near) maximally entangled states $\rho_{\rm out}$ [4]. As mentioned in the introduction, there are two classes of EPPs; 1-EPP and 2-EPP. Note that equivalence of a 1-EPP and a quantum error correcting code (QECC) was shown [5]. That is, for any (n, k) QECC, one can construct a 1-EPP.

2.3 Evaluation of EPPs

As evaluation factors of EPPs, the fidelity F between k Bell states $|\Phi^+\rangle^{\otimes k}$ and the remaining entangled states $\rho_{\rm out}$ after the EPP and the purification rate $R_{\rm P}$ of shared entangled states are known. F is defined as

$$F = {}^{\otimes k}\langle\Phi^+|\rho_{\rm out}|\Phi^+\rangle^{\otimes k}, \tag{4}$$

where $0 \le F \le 1$. The purification rate is defined as

$$R_{\rm P} = \frac{k}{n}P_{\rm S}, \tag{5}$$

where $P_{\rm S}$ is the success probability of the EPP. We refer to the limit $D = \lim_{n\to\infty} R_{\rm P}$ under $F \to 1$ as an *yield* of EPPs. The upper limit of the yield of 1-EPPs is equal to the quantum capacity from equivalence of a 1-EPP and a QECC.

3 Exponential bound on fidelity

In this section, we first explain an exponential bound on fidelity for quantum information transmission [11]. From equivalence of a 1-EPP and a QECC, the result can be directly converted to evaluation of 1-EPPs.

3.1 Quantum reliability function [11]

Let us survey the main result in Ref.[11], in which the concept of the classical reliability function is applied to quantum channels. As a difference from the classical theory, we use not an error prpbability but a fidelity F as an evaluation factor. Here, $1-F$ corresponds to the error prpbability. Suppose an (n, k) quantum error correcting code. Then the rate is $R = k/n$. The main theorem in Ref.[11] provides the lower bound of the minimum fidelity $F^\star_{n,k}$ with the best quantum error correcting code and is shown as follows.

Theorem 1 *[11]*
Let integers n, k and a real number R satisfy $0 \le k \le Rn$ and $0 \le R < 1$ (a typical choice is $k = \lfloor Rn \rfloor$ for an arbitrarily fixed rate R). Then for a memoryless channel $\mathcal{A}$, we have

$$F^\star_{n,k}(\mathcal{A}^{\otimes n}) \ge 1 - (n+1)^{2(d^2-1)} d^{-nE(R,P)} \tag{6}$$

where

$$E(R,\ P) = \min_Q [D(Q||P) + |1 - H(Q) - R|^+] \tag{7}$$

$|x|^+ = \max\{x, 0\}$, and the minimization with respect to Q is taken over all probability distributions on $\mathcal{X}$.

Here, $D(Q||P)$ and $H(Q)$ are well-known relative entropy and Shannon entropy, respectively. Since a qubit system is assumed in this paper, dimension d is always 2 and $\mathcal{X} = \{0, 1\}^2$.

3.2 Exponential bound on fidelity for EPPs

Since a 1-EPP is equivalent to a QECC, the bound shown in the previous subsection is applicable to an evaluation of 1-EPPs. Although tightness is an important factor to treat a *bound*, the bound is expected to be tight as far as a phase-damping channel is concerned because of its relationship to the random coding bound in the classical theory.

As for 2-EPPs, to derive a tight exponential bound on fidelity is not easy task. Therefore, we does not derive a tight bound for 2-EPPs but concentrate on verfying superiority of 2-EPPs to 1-EPPs. For this purpose, we follow the method in Ref.[8] in which effectiveness of feedback was demonstrated. In Ref.[8], a tight bound on error exponent without feedback is used, whereas error exponent by a specific code is used for a feedback scheme.

4 Superiority of a 2-EPP

In this section, we show superiority of a 2-EPP. As mentioned in Sec.3.2, we employ the exponential lower bound $E_2(R,P)$ of fidelity of a specific protocol as evaluation of a 2-EPP and employ exponential lower bound $E(R,P)$ of fidelity of the best protocol as evaluation of 1-EPPs. For this evaluation, 2-EPPs with variable number of initial shared entanglement may be desired. However, following the method in Ref.[5], we consider a 2-EPP consisting of a simple finite 2-EPP followed by an asymptotic 1-EPP such as one-way hashing. We employ the recurrence method [5] as the simple 2-EPP. Then we have a bound $E_2(R,P)$ by computing $E(R,P)$ of a 1-EPP whose input states are the outputs of the recurrence protocol.

4.1 Depolarizing channel

In this subsection, we compute $E(R,P)$ and $E_2(R,P)$ when a channel is assumed to be a depolarizing channel. Since the bound $E(R,P)$ is not necessarily tight for a depolarizing channel, computation in this subsection is regarded as preparations for the subsequent subsection. Figure 1 shows $E(R,P)$ (blue line) and $E_2(R,P)$ (red line) when $p = 0.2$. Here, p is defined in Sec.2.1.1. Since the yield of the 1-EPP by random codings is nothing [5],

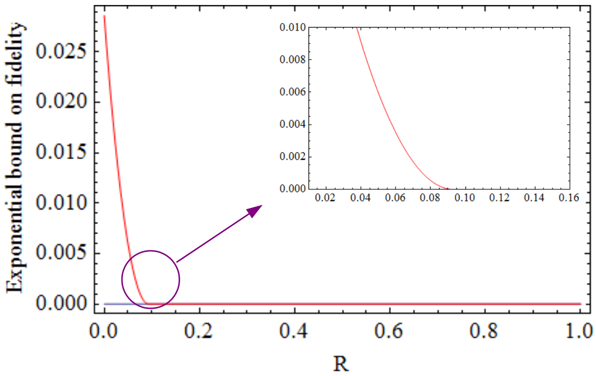

Figure 1: Exponential bound on fidelity for deporalizing channel (1). Case of $p = 0.2$.

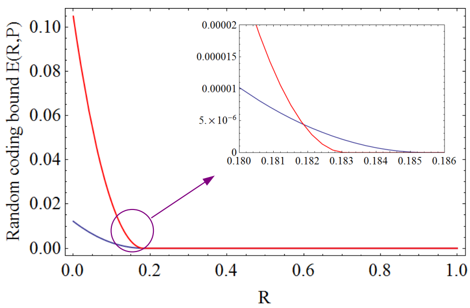

Figure 2: Exponential bound on fidelity for deporalizing channel (2). Case of $p = 0.142$.

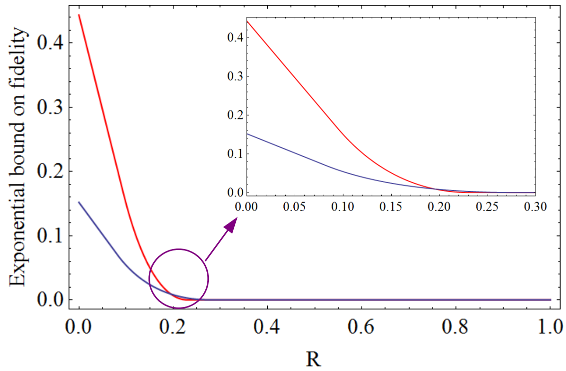

Figure 3: Exponential bound on fidelity for phase damping channel (1). Case of $p = 0.2$.

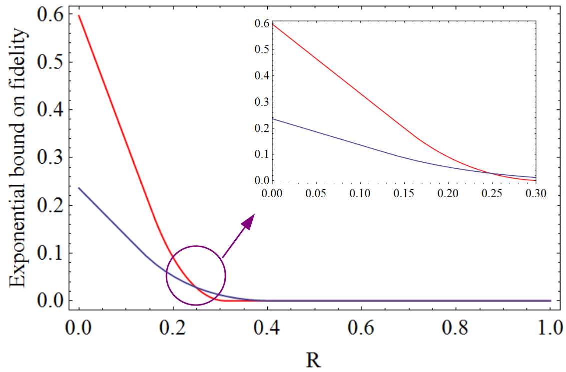

Figure 4: Exponential bound on fidelity for phase damping channel (2). Case of $p = 0.142$.

$E(R,P) = 0$ in this case. Therefore, clearly $E_2(R,P) > E(R,P)$ and we can reconfirm well-known superiority of 2-EPPs for depolarizing channel. Figure 2 shows $E(R,P)$ (blue line) and $E_2(R,P)$ (red line) when $p = 0.142$. In this case, the yield of the 1-EPP is higher than that of the 2-EPP [5]. However, $E_2(R,P) > E(R,P)$ when $R \lesssim 0.18$. This implies a possibility that the 2-EPP is superior to the 1-EPP at low rates, which is contrary to the result of yields.

4.2 Phase-damping channel

In this section, we show $E(R,P)$ and $E_2(R,P)$ when a channel is assumed to be a phase-damping channel. Since the bound $E(R,P)$ is expected to be tight for a phase-damping channel, the result in this subsection provides a clear conclusion. Figures 3 and 4 show $E(R,P)$ (blue line) and $E_2(R,P)$ (red line) when $p = 0.2$ and $p = 0.142$, respectively. Here, p is defined in Sec.2.1.2. Since a simple 2-EPP is performed before performing asymptotic 1-EPP, the yield of the 2-EPP is smaller than that of the 1-EPP. However, $E_2(R,P) > E(R,P)$ when $R \lesssim 0.2$ and it is concluded that a specific 2-EPP is superior to 1-EPPs at low rates.

5 Conclusion

In the present paper, we showed a property of an exponential bound on $1 - F$ for EPPs, which corresponds to the bound on error exponent in usual information transmission, when the initial shared entanglement is degraded by a depolarizing or a phase-damping channels. By comparing the exponential bound on $1-F$ of a specific 2-EPP to that of 1-EPPs, we see that the former is higher than the latter at low rates even if the yield of the 2-EPP is less than that of 1-EPPs. In particular, the result for the phase-damping channel clearly shows that a 2-EPP is superior to 1-EPPs and use of 2-EPP saves the number of initial shared entanglements even if there is only one kind of error. We will consider asymptotic 2-EPPs as shown in Ref.[19] and show its effectiveness for degraded entanglement by a phase-damping channel.

Acknowledgment: This work has been supported in part by JSPS KAKENHI Grant Number 24360151.

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