Quantum discord for two-qubit X states: A comprehensive study

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The study of correlations of bipartite systems has been invigorated over the last couple of decades. Various measures and approaches to demarcate the classical and quantum content of correlations have been presented. Entanglement has been the most popular of these correlations due to its inherent advantages in performing quantum computation and communication tasks [1]. Recently, however, motivated from a measurement perspective, other types of correlations have been proposed which try to capture the classical-quantum boundary [2, 3]. These include quantum discord, classical correlations, measurement induced disturbance, quantum deficit, and geometric variants of these measures. Of these, quantum discord has received enormous attention [4].

Quantum discord tries to capture quantum aspects of correlations beyond entanglement [5, 6]; there exist separable states which return a non-zero value of quantum discord. A recent avenue has been to try and find advantages of these correlations, both in the theoretical and experimental domain. For example, some interesting applications of quantum discord in quantum computation [7], state merging [8], remote state preparation [9], entanglement distillation [10], and representations of open quantum systems [11], have been reported.

Here we undertake a comprehensive study of the problem of quantum discord of the X-states of a two-qubit system. Several aspects of this problem has already been explored by a large number of authors [12]. The present study is largely provoked by the work of M. Ali, A. R. P. Rau, and G. Alber, Phys. Rev. A. 81, 042105 (2010), considered to be a seminal work on this problem (as indicated by the 300 plus citations in three years). Their group-theoretic argument is of an unusual kind. It goes like: *since the X-state problem has a particular symmetry, the optimal solution should necessarily be* **invariant** *under this symmetry.* [The more modest folklore wisdom has it that the solution to a problem just needs to be **covariant** under the symmetry of the problem]. With this invariance argument, Ali et al. *simply assert* that the optimal measurement for X-states is *always* the von-Neumann measurement along either x or z direction, it being understood that the conditional state of A post measurement on the system B commutes with σ_z, and that the conditional state of A under σ_x measurement is purer than that under σ_y measurement.

We can not make sense of this argument. We can, indeed, exhibit an X-state for which, despite the symmetry, neither the x nor z-projection is the best; perhaps more importantly, we can arrange both to return the **worst** value among **all** von-Neumann measurements. Several authors have more recently presented isolated examples wherein the Ali et al. assertion either fails or is doubtful, *numerically*; but the need for our comprehensive study originates in the fact that the Ali et al. argument is non-maintainable even in cases where their assertion somehow turns out to be numerically correct.

Our approach is geometric in nature, and we take advantage of methods involving notions like Stokes vectors, Poincare sphere, and Mueller matrix which have been in use in *classical polarization optics* for several decades [13], and we use symmetry to first bring the generic X-state problem to the simplest canonical form without loss of generality.

Correlations in a bipartite state ρ_{AB} are fully captured by the collection of all possible conditional states of A post measurements on B. For the two-qubit system, this collection is an ellipsoid [14] inside the Poincare sphere $\mathcal{P}$, and this is true for all ρ_{AB}: X-states are distinguished by the fact that the center of $\mathcal{P}$, the center of the ellipsoid, and $\mathrm{tr}_B\, \rho_{AB}$ are all *collinear*. This geometric, hence intrinsic, rendering and understanding of the 11-parameter family of X-states, in place of the 'shape' X of ρ_{AB} in the computational basis, proves central to our analysis.

Our geometric approach reproduces all known results, often in a much more economical manner, and underpin their geometric meaning. It leads also to several new results, and we mention a few :

- We identity a large class of states, which we call *circular states*, for which the optimal measurement and hence quantum discord are (perhaps surprisingly) as 'obvious' or manifest as those of Bell mixtures, even though $\mathrm{tr}_B \rho_{AB}$ is *not maximally mixed* for the circular states. This class can therefore be viewed as a generalization of Bell mixtures in respect of the present context.

- Some misconceptions in the literature [15] regarding X-states of vanishing quantum discord are clarified: X-states with two-way vanishing discord is a much larger family than the intersection of X-states of one-way vanishing discord, from A and B sides.

- A rigorous and geometric understanding as to why the optimal measurement for X-states *never* requires a four-element POVM (three-element POVM always suffices) is presented.

- Our analysis gives a clear understanding as to why Ali et al. assertion is numerically correct in a large portion of the parameter space of the family of X-states, notwithstanding their unusual argument.

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