Entanglement certainty from Heisenberg's uncertainty (arXiv:1305.3422)

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INTRODUCTION

Heisenberg's original formulation of the uncertainty principle considered sequential measurements of complementary observables, like position and momentum, performed on the *same* physical system, and the principle was that the second observable is *unavoidably disturbed* by the measurement of the first [1]. An alternative scenario considers *unavoidable uncertainty* related to the independent measurement of the two observables, with the measurements performed on two *distinct but identically prepared* quantum systems [2, 3].

The latter formulation of the uncertainty principle seems to receive more attention in modern times. For example, entropic uncertainty relations [4] typically capture this unavoidable uncertainty; consider a well-known example from Maassen and Uffink [5]. For any state ρ_S of a finite-dimensional quantum system S they find

$$H(X) + H(Z) \geqslant \log(1/c), \tag{1}$$

where $X = \{|X_j\rangle\}$ and $Z = \{|Z_k\rangle\}$ are any two orthonormal bases of $\mathcal{H}_S$, $H(X) := -\sum_j p(X_j)\log p(X_j)$ is the Shannon entropy associated with the probability distribution $p(X_j) := \langle X_j|\rho_S|X_j\rangle$ (similarly for $H(Z)$), and $c := \max_{j,k} |\langle X_j|Z_k\rangle|^2$ quantifies the complementarity between the X and Z observables. (Logarithms are taken in base 2 throughout.)

In [6] it was proven that an entropic uncertainty relation like (1) has a correspondent *entanglement certainty relation*. Ref. [6] considers the generation of entanglement between measurement devices and independent, although identically prepared, copies of some physical system, and proves that, when dealing with complementary observables, there is *unavoidable creation of entanglement* between at least one copy of the system and one measuring device.

MAIN RESULT

In this work, we offer a new point of view on what complementarity entails. As Heisenberg did originally, we consider sequential measurements performed on the same physical system, rather than idependent copies of the system; on the other hand, following [6–10], we focus on the entanglement generated between the system and the measurement devices. In general, for any X and Z, we can lower-bound the entanglement $E(X, Z)$ between the system and the measurement devices created from sequentially measuring X and Z with

$$E(X, Z) \geqslant \log(1/c), \tag{2}$$

where the c factor appearing here is *precisely the same c appearing in Eq.* (1). Here, we take E to be the distillable entanglement, i.e., the optimal rate to distill Einstein-Podolsky-Rosen (EPR) pairs using local operations and classical communication (LOCC) in the asymptotic limit of infinitely many copies of the state.

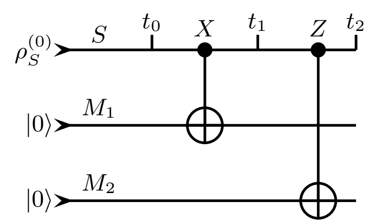

FIG. 1: Circuit diagram for the sequential measurement of the X and Z observables on system S.

Our approach relates in a novel way two basic concepts of quantum mechanics: complementarity—in the sequential-measurement scenario—and entanglement. Besides this fundamental interest, our results have direct operational interpretations. On one hand, they provide bounds on the usefulness of sequential bipartite operations—corresponding to the measurement interactions—for entanglement generation. On the other hand, we discuss below the application of our results to decoupling [11–15] and coherent teleportation [16, 17].

Figure 1 depicts the basic setup for our main result. At the initial time, denoted t_0, the system is in an arbitrary state $\rho_S^{(0)}$. It firsts interacts with device M_1, which measures observable X, and later it interacts with device M_2, which measures observable Z. The measurements are depicted with the controlled-NOT symbols although more generally they represent controlled-shift unitaries. We are interested in the bipartite entanglement $E(X, Z)$ between S and the joint system M_1M_2 at the final time, denoted t_2.

Fully complementary observables.—The case where X and Z are fully complementary, so-called mutually unbiased bases (MUBs), corresponds to $|\langle X_j|Z_k\rangle|^2 = 1/d$

for all j, k, and hence $c = 1/d$, where $d = \dim(\mathcal{H}_S)$. In the uncertainty relation (1), this leads to the maximum tradeoff in knowledge, with the r.h.s. becoming $\log d$, and hence perfect knowledge of X implies complete ignorance of Z. Likewise, in our main result, the r.h.s. of (2) becomes $\log d$. This implies that, *for any input state* $\rho_S^{(0)}$, sequentially measuring X and Z results in a maximally entangled state between S and M_1M_2. It may seem surprising that this is even true if we feed in a maximally-mixed state $\rho_S^{(0)} = \mathbb{1}/d$.

Partial complementarity.—Equation (2) also allows us to say that if X and Z are *almost* fully complementary, then for any input state, S is *almost* maximally entangled to M_1M_2 at time t_2. Furthermore, as long as there is some non-zero complementarity, i.e., $\log(1/c) > 0$, then there is guaranteed to be distillable entanglement at time t_2.

Proof of (2).—We refer to Ref. [18], where we offer two alternative proofs of (2). One approach is based on the uncertainty principle with quantum memory [19], applied at time t_1 in Fig. 1. The other approach invokes the monotonicity of entanglement under LOCC, which allows us to derive a slightly stronger version of (2) given in Ref. [18].

In what follows, we discuss the implications of our main result for decoupling and coherent teleportation, and also remark on generalisations of our results.

DECOUPLING

The correlations between two quantum systems can be destroyed, turning an arbitrary bipartite state $\rho_{SS'}$ into some tensor product $\sigma_S \otimes \sigma_{S'}$, with appropriate local operations. This idea, called decoupling [11–15] has specific applications in state merging [20] and quantum cryptography [21]. Our work identifies sequential complementary measurements as one such method to decouple. This is due to the monogamy principle: because system S is highly entangled to M_1M_2 at time t_2, then S cannot be too correlated with any other system S' at this time.

We make this precise by considering the relative entropy $D(\sigma\|\tau) := \mathrm{Tr}(\sigma \log \sigma) - \mathrm{Tr}(\sigma \log \tau)$. Letting $\rho_{SS'}^{(2)}$ denote the state of S and some other system S' at time t_2, we find that, for any initial state $\rho_{SS'}^{(0)}$,

$$D(\rho_{SS'}^{(2)} || \mathbb{1}/d \otimes \rho_{S'}^{(2)}) \leqslant \log(d \cdot c), \tag{3}$$

which is a corollary of Eq. (2). Indeed this implies that the final state $\rho_{SS'}^{(2)}$ is *almost* completely decoupled if the X and Z observables are *almost* fully complementary.

COHERENT TELEPORTATION

We have shown that the ability to produce entanglement and to decouple, using sequential measurements, is a quantification of the complementarity of those two measurements. It turns out there is a third perspective on complementarity. In the case when X and Z are MUBs, there exists a local unitary applied to M_1M_2 at time t_2 that recovers the input state $\rho_S^{(0)}$ on device M_1, i.e., we can "teleport" the input state of S to one of the measurement devices. This is commonly known as coherent teleportation [16, 17]. In this case, the channel $\mathcal{E}: S(t_0) \to S(t_2)$ is completely noisy, while the channel $\mathcal{E}^c: S(t_0) \to M_1M_2(t_2)$ is perfect. As we reduce the complementarity of X and Z, the channel $\mathcal{E}^c$ becomes less perfect, so we can consider the quantum capacity Q of $\mathcal{E}^c$, i.e., the optimal rate at which $\mathcal{E}^c$ allows for the reliable transmission of quantum information [22], as a measure of the complementarity of X and Z. We make this idea quantitative with the following bound

$$Q(\mathcal{E}^c) \geqslant \log(1/c),$$

which again is a corollary of our main result (2). This bound allows us to say that we can *approximately* teleport the state $\rho_S^{(0)}$ when X and Z are *almost* MUBs.

GENERALISATIONS

In Ref. [18] we consider two ways in which the above results can be generalised. The first considers measurement devices that are initially in mixed states instead of in the pure $|0\rangle$ states. While one expects mixed devices to be bad at accepting information, we find that a small amount of mixing does not completely ruin the entanglement created in the sequential measurements. Indeed we obtain a very simple generalisation of (2) when the devices are initially in mixed states.

An alternative generalisation considers the case where $n \geqslant 2$ measurement are done sequentially on the system. We find that the pairwise complementarity between any two successive observables provides a lower bound on the entanglement created. Again this gives a nice, simple generalisation of (2), to the case where many measurements are performed sequentially.

CONCLUSIONS

Summary.—Our work gives an alternative take on complementarity. Instead of discussing a trade-off of knowledge, as is typically done with uncertainty relations, we propose that a signature and a quantification of complementarity of two observables is given by the entanglement generated when the two observables are sequentially measured on the same system by means of a coherent interaction with corresponding measurement devices. We also offer the perspectives of decoupling and coherent teleportation.

The operational importance of complementarity has also been discussed by Renes and collaborators (see [23] and references therein); although we note that our physical scenario of sequential coherent complementary measurements is not obviously connected to mathematical theorems [24–27] based on knowledge or transmission of complementary information.

In general, we find it intriguing that the same complementary factor c appearing in uncertainty relations also appears in operational contexts. The fact that the complementarity of two observables measures their power to process quantum information in our scheme suggests to search for further "uncertainty" (or "certainty") relations for other information-processing tasks or quantum computing algorithms. Ref. [28] already made some progress along these lines, and we expect that our work will stimulate further results in the same perspective.

Appeal to AQIS audience.—Physicists who are familiar with the uncertainty principle should be engaged by our alternative view of complementarity, simply from a fundamental perspective. Information theorists who wish to better understand quantum information technologies should find it interesting that we have connected complementarity to the ability to accomplish several operational tasks, suggesting that complementarity is a useful resource, even when it is only *partial*.

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