An algebraic description of quantum nets

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Over the years the phase space description of quantum systems, as originally formulated by Wigner [1] in the context of quantum systems with one or more Cartesian degrees of freedom, has proved to be extremely fruitful in developing semi classical approximations in statistical mechanics, classical and quantum optics [2]. More recently, the Wigner description has provided the starting point towards characterizing Gaussian states [3] and in obtaining necessary and sufficient condition for entanglement in such states [4]. These developments have, in recent years, prompted vigourous activity aimed at developing similar descriptions for treating finite state quantum systems. A notable effort in this direction has been the work of Wootters and coworkers [5] who developed an elegant phase space description for N-state quantum systems for the case when $N = \mathfrak{p}^n$ where $\mathfrak{p}$ is a prime number. (The case of a general N is handled by taking tensor products appropriately). The phase space is taken to be an $N \times N$ grid where the coordinates q and momenta p take values in the finite field $\mathbb{F}_{\mathfrak{p}^n}$. The fact that the phase space variables take values in a field, albeit finite, endows the corresponding phase space with several geometric properties:

- The phase space has exactly $N + 1$ isotropic lines –'straight' lines through the origin.
- Each isotropic line gives rise to $N - 1$ lines parallel to it and thereby generates a striation of the phase space – decomposition of the set of N^2 phase points constituting the phase space space into N lines containing N points each. As there are $N + 1$ lines, one has $N + 1$ striations.
- Any two non parallel lines intersect at exactly one point and there are exactly $N + 1$ lines through a given phase point.

The 'classical net' comprising $N + 1$ striations, each containing N lines is then turned into a quantum net by associating a rank one projector with each line in a manner that is consistent with translational covariance. The requirement of translational covariance together with the geometrical properties listed above has the consequence that the projectors associated with lines in a striation are trace orthogonal and that the projectors associated with lines in different striations are mutually unbiased [6]. Phase point operators are then obtained from the projectors by adding up all the projectors corresponding to the lines passing through the chosen phase point and subtracting off the identity Each quantum net, set up in this manner, then defines a collection of N^2 phase point operators, one at each phase space point, which in turn leads to a possible definition of the Wigner distribution. Elementary considerations then show that the above construction leads to N^{N-1} distinct ways of associating a Wigner distribution with an N-state quantum system. Further, it is also clear that to construct these Wigner distributions one needs explicit knowledge of the $N + 1$ mutually unbiased bases.

In a recent work [7], we developed a procedure for setting up Wigner distributions in the original context based on what we call a Dirac inspired square root approach. In a later work [8] we showed how this approach could be fruitfully employed for obtaining a phase space description for N- state quantum systems. Here the underlying phase space in this work was taken to be a torroidal lattice – an $N \times N$ grid consisting of phase points (q, p) where the coordinates q and momenta p take values in the ring $\mathbb{Z}_N$, and we showed how one could set up Wigner distributions on such a phase space without introducing redundance in the description as was found necessary in some earlier treatments of this problem. The square root approach entails finding the square root of a certain matrix kernel which brings into play one undetermined sign at each phase point. Imposition of marginals conditions on isotropic lines fixes or relates these undetermined signs. Each choice of signs consistent with the marginals conditions then leads to a possible definition of Wigner distribution. For odd N, all the signs get uniquely fixed leading to a unique definition of the Wigner distribution akin to the original Wigner definition. In the N even case the situation turns out to be quite different. One finds that the marginals condition can not be imposed consistently on all isotropic lines but only on specific subsets thereof–orbits under $SL(2, \mathbb{Z}_N)$. The marginals property, restricted to the largest such subset, then leads to conditions on signs which are such that not all signs get fixed. Each choice for the set of free signs then leads to a possible definition of the Wigner distribution.

It is evident from the discussion above that both the approaches mentioned above lead to a multitude of definitions of Wigner distributions. In the quantum net approach this arises from the way vectors drawn from mutually unbiased bases are assigned to the lines while in our approach this is linked to the way the free signs are chosen. In the present work we investigate the question whether the two approaches are related. In particular, we apply our approach to the case treated by Wootters et al [5] i.e. the situation where $N = \mathfrak{p}^n$ and the cooordinates and the momenta take values in a finite field $\mathbb{F}_{\mathfrak{p}^n}$. Interestingly, we find that for $N = 2^n$ our approach seems to reproduce all the various definitions of Wigner distributions of Wootters et al purely algebraically without the explicit knowledge of the mutually unbiased bases in such dimensions. For odd prime powers, on the other hand, we recover only one of their definitions of Wigner distributions, the one that is Clifford covariant [9] and can be viewed as a direct descendent of that in the continuum case.

Our algebraic construction may find useful applications in investigating properties of quantum random access codes (QRAC) developed in [11] based on the full set of N^{N+1} phase point operators beyond $N = 8$.

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