Generalized monogamy of contextual inequalities from the no-disturbance principle

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Motivation and summary. It is known that we cannot assign a value to all observables in quantum mechanics. This feature has an observable consequence when we can perform several measurements and some can be jointly performed [1]. For example, suppose we have four measurements A, B, C, and D, out of which the pairs (A, B), (B, C), (C, D), and (D, A) can be jointly measured. The outcomes of a jointly performable measurements yield a joint probability distribution, e.g., we can define $p(a, b)$, where a and b refer to the outcomes of measurement A and B, respectively (we shall use lower case letters to indicate the outcomes of the respective measurements). Let each measurement return value 1 or -1. If the outcomes of all the measurements were determined prior to the time of measurement, the probability distributions $p(a, b)$, $p(b, c)$, $p(c, d)$ and $p(d, a)$ must satisfy [2]

$$\left| \sum_{a,b=1,-1} ab \cdot p(a, b) + \sum_{b,c=1,-1} bc \cdot p(b, c) + \sum_{c,d=1,-1} cd \cdot p(c, d) + \sum_{d,a=1,-1} da \cdot p(d, a) \right| \leq 2. \quad (1)$$

Such a bound exists for all Bell inequalities and is called a classical bound, because it is always satisfied in classical physics.

The Bell inequality of Eq. (1), known as the Clauser-Horne-Shimony-Holt (CHSH) inequality, can be violated up to $2\sqrt{2}$ in quantum mechanics [2] and applies to any physical setup where there are at least four dichotomic observables satisfying the joint measurablility mentioned above. Hence, for some physical system with more than four measurements, multiple CHSH inequalities can be defined. This does not imply, however, that all these CHSH inequalities can be violated simultaneously. In fact, simultaneous violation of certain combinations of CHSH inequalities is prohibited.

Not only for the CHSH inequality, correlation violating any Bell inequality is impossible in classical physics, thus a correlation violating a Bell inequality is a genuinely quantum one. The impossibility of the simultaneous violation implies that quantum correlation exhibits a monogamous behavior. Monogamy of quantum correlation for other Bell inequalities are also known [3] and found to be useful in secure quantum key distribution [4] and interactive proof systems [5]. Applications of monogamies go beyond these multi-party protocols [6, 7].

Despite its usefulness, the monogamy based on Bell inequalities is limited to when there are spacelike separated measurements. This limitation comes from the fact that Bell inequalities apply only to spacelike separated measurements. When measurements are not spacelike separated, quantum correlations can be identified by another type of inequalities known as *contextual inequalities* [8]. In this work, we prove that monogamy also exists for quantum correlations of general measurements by exploiting contextual inequalities. Our proof of monogamy is based on the principle of no-disturbance, which asserts that jointly performable measurements should not influence each other. We also formulate a necessary and sufficient condition for a set of measurements to admit monogamy using graph theoretic techniques.

Principle of no-disturbance. We now provide a mathematical formulation of the no-disturbance principle. Let us consider a physical system on which one can perform several different measurements A, B, C, etc. Let us assume that measurements A and B can be jointly performed as can measurements A and C. This implies the existence of the joint probabilities $p(a, b)$ and $p(a, c)$. The principle of no-disturbance is then the condition that the marginal probability $p(a)$ calculated from $p(a, b)$ is the same as that calculated from $p(a, c)$, i.e.,

$$\sum_b p(A = a, B = b) = \sum_c p(A = a, C = c) = p(A = a). \tag{2}$$

Monogamy of KCBS-type inequalities. We concentrate first on the simplest contextual inequality, the Klyachko-Can-Binicoglu-Shumovsky (KCBS) inequality, [8] that was introduced to test a nonclassical feature of a single (three-level) system and construct a monogamy relation for it. Similar monogamies hold for any inequalities of this kind [9]. The KCBS inequality reads

$$\sum_{i=1}^{5} p(A_i = 1) \leq 2, \tag{3}$$

where A_i are measurements with outcomes $a_i = 0, 1$. These measurements are cyclically compatible (i.e., it is possible to experimentally determine $p(a_i, a_{i+1})$ (where one identifies a_6 with a_1)) and exclusive (i.e., $a_i a_{i+1} = 0$). These measurements can be represented by the "commutation graph" corresponding to a pentagon where the vertices of the pentagon graph represent the five measurements and edges between any two vertices indicate that the two corresponding measurements can be jointly performed and are mutually exclusive (see the pentagons in Fig. 1). Analogous inequalities can be constructed for larger number of measurements as well [9, 10].

Let us derive a monogamy relation for the KCBS contextual inequality from the no-disturbance principle. Consider two sets of cyclically compatible and exclusive measurements $\{A_i\}$ and $\{A'_i\}$. Each set gives rise to a KCBS inequality (3). Let us assume that the triple A_1, A'_1, A'_2 are jointly measurable and mutually exclusive, as is also the triple A_4, A_5, A'_5. This scenario is represented by the commutation graph in Fig. 1. Therefore, in addition to $p(a_i, a_{i+1})$ and $p(a'_i, a'_{i+1})$, one can experimentally determine probabilities $p(a_1, a'_1, a'_2)$ and $p(a'_5, a_4, a_5)$. This condition is similar to a condition imposed in the derivation of Bell monogamies, namely that a common observer chooses the same settings for the violation of Bell inequalities with all other observers.

We introduce the no-disturbance principle (2) by setting $p(A_1 = 1) = p$ and $p(A'_5 = 1) = q$. Mutual exclusiveness implies that $p(A'_1 = 1) + p(A'_2 = 1) \leq 1 - p$ and $p(A_4 = 1) + p(A_5 = 1) \leq 1 - q$ in addition to $p(A_i = 1) + p(A_{i+1} = 1) \leq 1$ and $p(A'_i = 1) + p(A'_{i+1} = 1) \leq 1$. However, this already implies $\sum_{i=1}^{5} p(A_i = 1) \leq 2 - q + p$ and $\sum_{i=1}^{5} p(A'_i = 1) \leq 2 - p + q$ and therefore the monogamy relation

$$\sum_{i=1}^{5} p(A_i = 1) + \sum_{i=1}^{5} p(A'_i = 1) \leq 4 \tag{4}$$

holds. Therefore, only one KCBS inequality out of the two sets $\{A_i\}$ and $\{A'_i\}$ can be violated in all theories that obey the no-disturbance principle such as quantum mechanics.

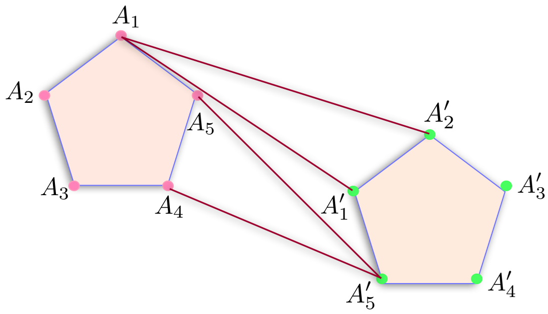

FIG. 1: Graphical representation of two KCBS inequalities that satisfies the monogamy relation.

Having illustrated the method for deriving monogamy relations for contextual (and Bell) inequalities we now proceed to formulate it using some graph-theoretic notions. To do so, we first state the following Proposition 1 whose proof is provided in the technical version of the work [11].

Proposition 1: "A commutation graph G representing a set of n measurements (for any n) admits a joint probability distribution for these measurements if it is a chordal graph."

A chordal graph is a graph that does not contain an induced cycle of length greater than 3, i.e., each of the cycles of four or more vertices in the graph must have a chord, an edge connecting two non-adjacent vertices in the cycle. This class of graphs comprises a large class of all graphs of n vertices and the above Proposition 1 excludes the construction of contextual inequalities (or Kochen-Specker proofs) from all such graphs.

Note that while many contextual inequalities involve rank-1 projectors, where the edges of the graph denote mutual exclusiveness in addition to compatibility, this assumption is not crucial to the derivation of monogamies. This can be seen from the derivation of the Bell inequality monogamies, where only compatibility is required.

We now proceed to explicitly identify the commutation graphs that give rise to monogamy relations for a given set of n KCBS-type contextual inequalities (with classical bound R). This is done in the Proposition 2 which provides the necessary and sufficient condition for a commutation graph to give rise to a monogamy relation using the method outlined above, its proof is provided in the technical version of the work [11].

Proposition 2: "Consider a commutation graph representing a set of n KCBS-type contextual inequalities each of which has classical bound R. Then this graph gives rise to a monogamy relation using the outlined method if and only if its vertex clique cover number is $n * R$."

The vertex clique cover number is the minimal number of cliques required to cover all the vertices of the graph. The above Proposition can be extended to the case when when one is interested in the monogamy of a set of n_k different contextual inequalities with different classical bounds R_k, with $\sum_k n_k = n$. Then the condition becomes that the vertex clique cover number equal $\sum_k n_k R_k$. This gives a very powerful method of identifying whether a given graph exhibits contextual monogamy.

The argument so far proves that the monogamy relation (4) exists provided that there are measurements obeying the constraints of mutual exclusiveness and joint measurability as required by the commutation graphs. In the technical version of the work, we show an explicit construction of a set of projectors in a four-dimensional real space that meets these constraints [11].

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