Continuous Variable Entropic Uncertainty Relations in the Presence of Quantum Memory

Mario Berta[1], Matthias Christandl[1],Fabian Furrer[2],Volkher Scholz[1],Marco Tomamichel[3]

[1]Institut für Theoretische Physik, ETH Zürich, 8093 Zürich
[2]Department of Physics, Graduate School of Science,
University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
[3]Centre for Quantum Technologies, National University of Singapore, Singapore 117543

June 17, 2013

Abstract

We generalize entropic uncertainty relations in the presence of quantum memory [Nature Physics 6 (659), 2010], and [Physical Review Letters 106 (110506), 2011] in two directions. First, we consider measurements with a continuum of outcomes, and, second, we allow for infinite-dimensional quantum memory. To achieve this, we introduce conditional differential entropies for classical-quantum states on von Neumann algebras, and show approximation properties for these entropies. As an example, we evaluate the uncertainty relations for position-momentum measurements, which has applications in continuous variable quantum cryptography and quantum information theory.

Introduction Uncertainty relations play a fundamental role in quantum mechanics. A first uncertainty relation was discussed by Heisenberg [14] by analyzing the disturbance induced by a position measurement with a certain resolution on a subsequent momentum measurement. Shortly after Kennard [16] and Robertson [24] introduced an uncertainty relation in which the product of the standard deviation of the distributions of two non-commuting observables applied to the same state is bounded from below. These uncertainty relations were then enhanced by replacing the standard deviations by entropies leading to so-called entropic uncertainty relations. For position and momentum operators they were first studied by Hirschman [15], and subsequently improved by Beckner [1] and Bialynicki-Birula and Mycielski [5]. Deutsch stated in [10] an entropic uncertainty relation for finite-dimensional observables, which was tightened by Maassen and Uffink [19] following a conjecture of Kraus [17],

$$H(X) + H(Y) \geq \log \frac{1}{c} , \tag{1}$$

where $H(X)$ and $H(Y)$ are the Shannon entropies of the outcome distributions of non-degenerate measurements X and Y and $c = \max_{i,j} |\langle x_i | y_j \rangle|^2$ with $|x_i\rangle$ and $|y_j\rangle$ the eigenvectors of X and Y, respectively. The inequality was further generalized to observables described by positive operator valued measures, and to different entropies[1] (see the recent review articles [32, 6] for references).

But so far, the connection of uncertainty relations to another fundamental quantum feature, entanglement, was not fully understood. The discussion already started in the famous EPR paper in 1935 [11], but a quantitative and operationally useful criteria was missing. It was only recently realized that uncertainty should not be treated as absolute, but with respect to the

[1]In [19], the inequality was already shown for general α-Rényi entropies [23].

knowledge of an observer [13, 21, 2]. And if the side-information of the observer is quantum, one obtains a subtle interplay between the observed uncertainty and the entanglement between the measured system and the quantum side-information. This can be quantified by an entropic uncertainty relation stated by conditional entropies [7, 21, 2, 30, 9, 8]. For a bipartite state ρ_{AB} and measurements as above, we have [2]

$$H(X|B) + H(Y|B) \geq \log\frac{1}{c} + H(A|B)\ . \tag{2}$$

Here, $H(X|B)$ and $H(Y|B)$ is the conditional von Neumann entropy of the measurement outcomes of X and Y given the side information B,[2] and $H(A|B)$ is the conditional von Neumann entropy of the state ρ_{AB}. The latter quantity can be negative in the presence of entanglement, and measures the initial correlations between A and B. Because of the monogamy property of entanglement, the tripartite scenario allows a particularly elegant formulation [2, 21]: it holds for any tripartite quantum state ρ_{ABC} and measurements as above,

$$H(X|B) + H(Y|C) \geq \log\frac{1}{c}\ . \tag{3}$$

Note that the constant c is the same as in inequality (1) and the tripartite formulation can be seen as a further generalization of the one shown by Massen and Uffink.

The entropic uncertainty relations with quantum side information have various applications in quantum information theory. Most prominently, the tripartite version in (3) can be used as a straightforward tool to prove security against arbitrary (coherent) attacks of certain quantum key distribution protocol [2, 29]. For that purpose, the uncertainty relation stated in equation (3) in terms of the von Neumann entropy has been extended to the smooth min- and max-entropies [30], which quantify the extractable key length in the one-shot scenario. For an overview about the smooth min- and max-entropies we refer to [22, 31]. Recently, the tripartite uncertainty relation has been generalized furrther to an entire class of entropies including von Neumann and min-/max-entropy [8].

However, all of the previously mentioned results involving quantum side information assume quantum systems with finitely many degrees of freedom. A first generalization of the tripartite uncertainty relation to infinite-dimensional quantum systems has been derived for the smooth min- and max-entropy by some of the authors in [3]. While the quantum side-information could be arbitrary, only measurements withe a finite-number of outcomes were considered. Based on this uncertainty relation, the first quantitative security analysis of a continuous variable quantum key distribution protocol against arbitrary attacks has been presented in [8]. The extension to infinite number or continuous outcomes was recently also addressed in [12] where entropic uncertainty relations with quantum side information in terms of the von Neumann entropy were derived, which also apply to continuous position-momentum measurements. Yet, they only consider finite-dimensional quantum side information.

Results. In this work, we present tripartite entropic uncertainty relations with quantum side-information for infinite-dimensional quantum systems without restrictions on the observables and the quantum side information. The uncertainty relations are derived for the von Neumann entropy as well as the (smooth) min- and max-entropy. Note that due to the fact that the outcomes can be continuous, we have to use differential conditional entropies.

The proof strategy we are employing is to first introduce differential conditional von Neumann $h(X|B)$ and differential conditional min- and max-entropy, $h_{\min}(X|B)$ and $h_{\max}(X|B)$, for classical systems X described by a measure space and quantum side-information B modeled

[2]More precisely, $H(X|B)$ is the von Neumann entropy of the post-measurement state $\rho_{XB} = \sum_i (|x_i\rangle\langle x_i|_A \otimes \mathbb{1}_B)\, \rho_{AB}\, (|x_i\rangle\langle x_i|_A \otimes \mathbb{1}_B)$.

by an observable algebra.[3] Intuitively, continuous classical systems may be thought of as being approximated by discrete systems in the limit of infinite precision. Hence, we expect that reasonable defined differential entropic quantities have a similar behavior. We make this precise by proving that the differential conditional entropies $h(X|B)$, $h_{\min}(X|B)$ and $h_{\max}(X|B$ can be approximated by their discretized counterparts. In particular, if X is a classical system with outcome range being the real line, and X_δ its restriction to a covering of $\mathbb{R}$ by half open intervals of length δ, then

$$h(X|B) = \lim_{\delta \to 0} \big(H(X_\delta|B) + \log \delta \big) \tag{4}$$

if the differential entropy $h(X|B)$ is finite. A similar result is derived for the differential min- and max-entropies $h_{\min}(X|B)$ and $h_{\max}(X|B)$.

The tripartite uncertainty relation for measurements with a continuous outcome range are then derived by means of these approximation results from the discrete outcome case. In the discrete case we follow a similar strategy to the one in [8], where the tripartite uncertainty relation is derived from some elementary properties of the entropies.[4] Among these properties is the duality which says that $H(A|B) = -H(A|C)$, whenever the state of the joint system ABC is pure. This property turned out to be very important in connection with tripartite uncertainty relations and is here proven for our general definition of a conditional von Neumann entropy.

The derived uncertainty relation holds in the general case where the measurements are general positive operator valued measures on an appropriate measure spaces allowing sufficiently nice partitions. The inequality reads exactly like in equation (2) exept that the entropies are exchanged by their differential versions and the overlap is computed by a limit along finer and finer partitions.

We then analyzed the uncertainty relation for position and momentum measurements in the case of finite and infinite precision measurements. We start in the case of a finite measurement resolution which we model by a binning of the outcome range into intervals of length δ. In that case the overlap term only depends on δ and can be computed [28]. The behavior for small δ is given by $c(\delta) \approx \delta^2 \backslash (2\pi)$. We then show that the obtained uncertainty relation in terms of the min- and max-entropy is tight even without side-information. In particular, a pure state which has only support on one interval of the measurement for which the max-entropy is evaluated achieves equality. The tightness question is more subtle in the case of the von Neumann entropy. There we can only show that in the case of trivial side information equality cannot be achieved for small δ.

In the case of infinite precision measurements, and thus, continuous outcomes the constant can be obtained by taking the limit for $\delta \to 0$. The resulting inequality is then given by

$$h(Q|B)_\omega + h(P|C)_\omega \geq \log 2\pi,$$

which generalizes previous results in [20, 4, 25, 26, 27]. It turns out that the uncertainty realtion is again tight for the min- and max-entropy without side-information. In the case of the von Neumann entropy the inequality cannot achieve equality without side-information. But it is still open if in the case with side-information (which enhances the inequality) the uncertainty relation can be tight.

[3] The definition of the conditional von Neumann entropy with infinite-dimensional systems generalizes the one recently given in [18].

[4] In the case of the min- and max-entropy the inequality can also be obtained from the one in [3] for finite numbers of outcomes. But this strategy is not applicable to the von Neumann entropy.

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