Universal implementation of energy eigenbasis measurement

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We show a scheme to universally implement a projective measurement in the energy eigenbasis on a system evolving by an unknown Hamiltonian H based on the phase estimation algorithm. To apply the phase estimation algorithms for unknown Hamiltonian systems, two new algorithms are introduced. One is for asymptotically but universally implementing a controlled-unitary operation $C_{U(t)}$ of a unitary operation $U(t) = e^{-iHt}$ up to the global phase of $U(t)$ for an unknown Hamiltonian H using dynamical decoupling. Another is a new deterministic quantum computation with one pure qubit (DQC1) algorithm for evaluating the absolute value of the trace of $U(t)$ without using $C_{U(t)}$. We analyze accuracy of the implementation of $C_{U(t)}$ by using this DQC1 algorithm. The DQC1 algorithm is also used for removing the ambiguity due to the periodicity of the phase and the effect of the global phase to obtain the energy eigenvalue as an outcome.

I. MOTIVATION AND SUMMARY

One of the hallmarks of quantum mechanics is that a measurement operation generally changes the state of the measured system. Especially, a projective measurement of an observable sets the measured system in an eigenstate of the observable. Let us consider the case where a projective measurement is performed on a system evolving according to its own internal Hamiltonian and the internal Hamiltonian is chosen for the observable of the projective measurement. In this case, the system will be projected onto an energy eigenstate, which can be useful to stabilize the system as energy eigenstates are stationary states, and is also important for experimental confirmation of the recent theories on microscopic origins of thermodynamical relations [1–3]. The system left alone, however, evolves according to a unitary evolution determined by the internal Hamiltonian. Hence, the implementation of the energy eigenbasis projective measurement requires manipulation from the outside of the system, which is possible if there is another system on which we can implement any quantum map at our will. A system with such high controllability is a quantum computer.

Using a quantum computer, it is a trivial task to implement the energy eigenbasis projective measurement if the internal Hamiltonian is already identified. Simply, we transfer the state of the system into the quantum computer, perform the energy eigenbasis projective measurement within the quantum computer, and return the resulting state back to the system. The resource required for this method largely depends on the implementation of the projective measurement within the quantum computer. A straightforward implementation is to use a unitary transformation that maps the energy eigenbasis to the computational basis and perform the projective measurement in the computational basis. A realization of a such unitary transformation must first compute the eigenvectors of the Hamiltonian, which makes this naive method extremely inefficient. A better implementation involves Kitaev's *phase estimation algorithm* [4] and is known to provide a polynomial implementation of the energy eigenbasis projective measurement for local Hamiltonians [5, 6].

On the other hand, for unknown Hamiltonians, the schemes presented above (with or without the phase estimation algorithm) are inapplicable. Brute-force methods to estimate unknown Hamiltonians, such as process tomography [7], require time resource of $O(d^2)$, where d is the dimension of the system, hence increases exponentially in the number of systems.

One proposal assumes that the input is encoded in a particular subspace and that there is another subspace on which the Hamiltonian acts as the identity operator [8]. In this case, the phase estimation algorithm can be applied without estimating the Hamiltonian. The assumptions are satisfied in particular setups such as in linear optical quantum computation using photon qubits, but cannot be generally applied to other settings

In this paper, we propose a scheme to implement the energy eigenbasis measurement on a system evolving under an unknown, general Hamiltonian and achieve exponential speedup over the brute-force method. In fact the running time is independent of the dimension of the system. We also analyze the accuracy of the scheme and prove the running time to be exponential in the accuracy of the measurement, hence confirm that our scheme respects the computational complexity hierarchy.

Our scheme is based on two new subroutines. The first implements an approximated operation of the controlled unitary evolution operator by using the dynamical decoupling method [9]. The other evaluates a quantity needed to run the first algorithm and is based on the deterministic quantum computation with one pure qubit (DQC1) algorithm [10].

II. PHASE ESTIMATION ALGORITHM

The phase estimation algorithm is a quantum algorithm to estimate the phase factor $0 \leq \theta_i < 2\pi$ of the eigenvalue $e^{i\theta_i}$ of a unitary operation U, when an eigenstate $|\theta_i\rangle$ is given as an input state. The algorithm uses controlled-unitary operations of $U, U^2, U^{2^2}, \cdots, U^{2^N}$ where N denotes the number of control qubits. A controlled-unitary operation C_U of an unitary operation U is defined by

$$C_U := |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes U \tag{1}$$

on $\mathcal{H}_c \otimes \mathcal{H}_t$ where the Hilbert spaces of the control system and the target system are represented by $\mathcal{H}_c = \mathbb{C}^2$ and $\mathcal{H}_t = \mathbb{C}^d$, respectively.

As N increases, the probability to obtain an outcome outside a fixed range of an eigenvalue decreases exponentially in terms of N, whereas the total calling time of U increases exponentially. If we apply the phase estimation algorithm to an arbitrarily superposed input state $|\phi\rangle = \sum_i \alpha_i |\theta_i\rangle \in \mathcal{H}_t$ where $\sum_i |\alpha_i|^2 = 1$, the algorithm implements a projective measurement $\{|\theta_i\rangle\langle\theta_i|\}$ on $\mathcal{H}_t$ as $N \to \infty$.

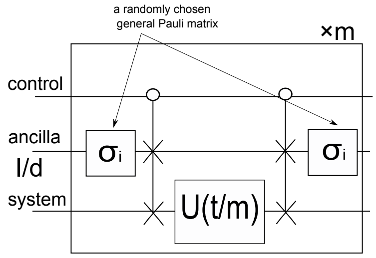


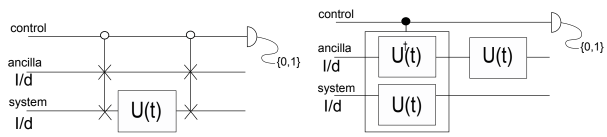
FIG. 1. A quantum circuit representation of the algorithm approximately implementing $C_U(t)$ on the controlled and target systems. The two generalized Pauli operations σ_i in a sequence are identical but they have to be chosen randomly for each iteration.

III. UNIVERSAL CONTROLLIZATION

We consider to perform the energy eigenbasis measurement on an *unknown* Hamiltonian H using the phase estimation algorithm. The controlled unitary operations $C_U, C_{U^2}, \cdots C_{U^{2^n}}$ are required to run the algorithm. We propose an algorithm that *asymptotically* and *universally* implements a controlled unitary operation when the unitary operation is given as $U(t) = e^{-iHt}$ for an *unknown* Hamiltonian H, without identifying the description of H by Hamiltonian tomography.

The algorithm is the following. We add an ancilla system where its Hilbert space is represented by $\mathcal{H}_a = \mathbb{C}^d$ and its initial state is prepared in a maximally mixed state $\mathbb{I}/d$. We divide the time evolution $U(t)$ on $\mathcal{H}_t$ into m iterations of $U(\frac{t}{m}) = e^{-iH\frac{t}{m}}$, and insert Fredkin gates and randomly chosen general Pauli operations before and after each $U(\frac{t}{m})$ as shown in Fig. 1. The random Pauli operations causes the dynamical decoupling effect between the ancilla system and the other systems. Thus the undesired entanglement between the ancilla system and the other systems is weakened and m iterations of this procedure $\Gamma^m_{U(t/m)}$ implements a map that is an approximation of $C_{U(t)}$.

The difference between the maps $\mathcal{C}_{U(t)}$ and $\Gamma^m_{U(t/m)}$ can be evaluated in terms of the dia-



(a) The new DQC1 algorithm

(b) A different circuit with the same action

FIG. 2. Quantum circuits representing the DQC1 algorithm for evaluating $|\mathrm{Tr}U(t)|^2$.

mond norm [11] by

$$\left\|\mathcal{C}_{U(t)}^{(m\varphi_{U(t/m)})} - \Gamma_{U(t/m)}^{m}\right\|_{\diamond} = 1 - (a_{U(t/m)})^m \quad (2)$$

where

$$a_{U(t/m)} := |\mathrm{Tr}[U(t/m)]/d|. \quad (3)$$

The factor $a_{U(t/m)}$ can be evaluated by

$$\begin{aligned} a_{U(t/m)} &= O[((\mathrm{Tr}H)^2 - \mathrm{Tr}H^2)t^2/(dm)^2] \\ &\leq 1 + O[(\Delta E_{\max})^2 t^2/m^2], \end{aligned} \quad (4)$$

where $E_i = -\theta_i m/t$ is an eigenvalue of H corresponding to the eigenstate $|\theta_i\rangle$ and $\Delta E_{\max}$ is the maximum energy difference (the largest eigenvalue minus the smallest eigenvalue) of H. Thus the right hand side of Eq. (2) can be bounded by $[(\Delta E_{\max})^2 t^2/m]$.

IV. EVALUATING $\mathrm{Tr}U(t)$ WITHOUT CONTROLLIZATION

We need to choose $(a_{U(t/m)})^m$ high enough for universal controllization. Namely, we need to evaluate the value of $|\mathrm{Tr}U(t/m)| = |\mathrm{Tr}e^{-iHt/m}|$ for unknown H. We present a new DQC1 algorithm evaluating $|\mathrm{Tr}U(t)|$ for an arbitrary t by using the relationship $|\mathrm{Tr}(U(t) \otimes U^\dagger(t))| = |\mathrm{Tr}U(t)|^2$. The algorithm is represented in the quantum circuit shown in Fig. 2. Note that only the measurement in the computational basis is required in our algorithm, since we only need to know the absolute value of $\mathrm{Tr}U(t)$.

This algorithm can be also used for evaluating the maximum energy gap of the unknown Hamiltonian. We prove that the probability of finding $(a_{U(t)})^2$ greater than $1/2$ is exponentially small in terms d for $\Delta Et > 4\pi$, if we consider Hamiltonians randomly sampled from the Circular Unitary Ensemble. By repeating the evaluation of the value of $(a_{U(t/2^k)})^2$ for $k = 2, 3, \cdots$, we estimate $\Delta E_{\max}$ by searching k' such that $(a_{U(t/2^{k'})})^2 \geq 1/2$ using the DQC1 algorithm.

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