A gambling interpretation of some quantum information-theoretic quantities [a] [b]

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It is known that repeated gambling over the outcomes of independent and identically distributed (i.i.d.) random variables gives rise to alternate operational meanings of entropies in the classical case in terms of the doubling rates. We give a quantum extension of this approach for gambling over the measurement outcomes of tensor product states. Under certain parameters of the gambling setup, one can give operational meaning of von Neumann entropies. We discuss two variants of gambling when a helper is available and it is shown that the difference in their doubling rates is zero for classical roulettes and is the quantum discord for quantum roulettes. Lastly, a quantum extension of Kelly's gambling setup in the classical case gives a doubling rate that is upper bounded by the Holevo information.

Quantum information theory [1–3] deals with the information content in the quantum systems and is a generalisation of classical information theory (see Ref. [4] for example) for quantum systems.

The measurement outcome of a quantum system is a random variable and the measurement alters the quantum state in general. We confine ourselves to finite dimensional Hilbert spaces that describe the quantum states and a probability mass function would describe the measurement outcome random variable that can be computed using the postulates of quantum mechanics.

If after a measurement, a quantum system is prepared again in the same state as before the measurement and the same measurement process is repeated, the sequence of the measurement outcomes is a sequence of i.i.d. (independent and identically distributed) random variables.

If bets are placed on the measurement outcomes of a quantum system, then the apparatus becomes a gambling device or a quantum roulette. Quantum gambling has been studied before in different contexts [5–8]. We note that none of the these works study the log-optimal gambling strategies, which, on the other hand, has been studied well for the classical case (see for example [4, 9, 10] and references therein).

Quantum systems exhibit certain characteristics that are not possible classically. Bell inequalities [11, 12] give classical limits to the figure of performance for certain setups and these inequalities could be violated by quantum systems. We show that the quantum gambling devices too exhibit certain characteristics that are impossible to replicate classically.

At an information-theoretic level, the joint von Neumann entropy of the quantum systems A and B can be smaller than the von Neumann entropy of the system B alone giving rise to negative conditional entropies. This is, as is well known, impossible for classical Shannon entropies.

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[b] A fuller version of this paper shall appear in the http://arxiv.org/list/quant-ph/recent listing on Tuesday, June 18, 2013. The arxiv identifier is not available yet.

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Kelly defined a log-optimal gambling strategy by applying the law of large numbers can be applied to the logarithm of the factor (a random variable) by which Alice's wealth grows in a gamble. Thus, one can loosely claim that Alice's wealth is an exponential function of the number of gambles [9]. (We define this more precisely later.) This approach has been developed further with the side information (or a helper) in Ref. [4].

The exponent is a function of payoffs that the casino owner, Charlie, offers for each outcome, outcome probability distribution, and Alice's strategy. When we optimise the strategy under certain conditions, then the entropy (Shannon or von Neumann) appears in the exponent.

We note that these entropies (and certain information measures) have deep operational interpretations in classical and quantum information theory (see Ref. [1] and references therein).

In the classical case, Alice chooses how the wealth with which she is gambling is going to be distributed across the various outcomes. As an example for two outcomes, Alice could bet half of her money on each of the outcomes.

For the quantum case, Alice can additionally make a choice of the measurement operators. Any classical roulette is a special case of a quantum roulette. Under certain conditions, for the quantum roulette, the conditional entropy appears in the exponent mentioned above and since the quantum conditional entropy can be negative, it implies that Alice's wealth could potentially increase more than the classical roulette.

We also consider a case when a helper named Bob is available for the gambler to make more money. Bob has access to a quantum system that is correlated with Alice's quantum system. Bob is broke and has no money to gamble on his system. Bob offers Alice help in two ways.

In the first variant, Bob the helper reports the measurement outcome to Alice who now knows the collapsed state of her quantum system and uses this information to further optimise her exponent (or the doubling rate).

In the second variant, Bob leases out his quantum system to Alice who then gambles on the composite quantum system consisting of her and Bob's systems. In return, Bob demands a share in Alice's accrued wealth and wants Alice to retain the portion of wealth that Alice would have accrued had she ignored the correlations between the two systems. Bob's argument is that Alice can win more by taking the correlations into account and it thus a win-win situation for both him, since he earns money after being broke, and Alice since she earns more money.

Under certain conditions, for the classical gambling, these two variants give rise to the same doubling rates whereas, for quantum gambling, they give rise to different doubling rates whose difference is equal to the quantum discord [13, 14], a quantity that has been studied in a completely different context.

Quantum discord is interpreted as purely the quantum part of the total correlations between the two quantum systems. That these two variants are the same classically in terms of the doubling rate lends support to the above interpretation.

Kelly gave another interesting interpretation of the mutual information [9]. Suppose Alice, knowing the output of a communication channel, bets on the inputs to the channel, then under certain conditions, the optimised doubling rate is equal to the mutual information. If one extends Kelly's result for the quantum case, one gets a doubling rate in terms of a certain mutual information that is a function of the measurement operators and, using the

Holevo bound, is upper bounded by the Holevo information.

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