Local Hidden Variable Theoretic Measure of Quantumness of Mutual Information

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Abstract. The concept of classicality of quantum states originates from the local hidden variable (LHV) theory. It is invoked expilictly in chracterizing quantumness of correlations between observables of the subsystems of a system but not by the existing measures of quantumness of mutual information. Here, based explicitly on the considerations of LHV, a measure of quantumness of mutual information, $Q_{\rm LHV}$, in a system of two spin-1/2 particles is introduced. It does not require optimization of the classical information and turns out to be identical with the measurement induced disturbance when the Bloch vector of each of the spins is non-zero. However, whereas the measurement induced disturbance does not provide unique measure when the Bloch vector of one or both the spins is zero, the $Q_{\rm LHV}$ is determined uniquely even in those cases. The $Q_{\rm LHV}$ can be evaluaed analytically exactly if the Bloch vector of both the spins is zero and turns out to be the same as the symmetric discord. If the Bloch vector of only one of the two spins vanishes then $Q_{\rm LHV}$ is found to be the same as the quantum discord for certain states.

Keywords: Local Hidden Variables, Quantum Discord, Measurement Induced Disturbance, Symmetric Discord

Entanglement, a manifestation of quantumness of correlations between observables of the subsystems of a composite system, once thought to be an essential ingredient for distinctive quantum features in quantum information processing, is no longer considered to be so as it is found that the unique features of quantum information processing are contained in the quantum nature of information which does not necessarily require entanglement (see [1] and references therein). The characterization of the quantumness of information and that of the correlations between the observables are based on different cosiderations: Whereas quantumness of correlations between observables of parts of a composite system is characterized by their incommensurability with the predictions of the local hidden variable (LHV) theory (see [2] and references therein), the quantumness of information is based on identifying classical content of mutual information in the quantum state [3]-[6]. Different protocols for identifying classical content of information lead to different measures of quantumness of information like quantum discord [3], quantum deficit [4], measurement induced disturbance [5], symmetric discord [6] and others [1]. These measures show that even a separable state may contain quantum features in its information content.

Since the approaches to identify the quantumness of correlations between the observables explicitly invoke LHV theory but the measures of quantumness of information do not invoke that theory explicitly, in the interest of a unified approach, it is arguably desirable to characterize quantumness in different properties based on same considerations. In view of the fact that the concept of classicality of a quantum state stems from the LHV theory, like the characterization of correlations in the observables, the LHV theoretic characterization of quantumness of information would be the natural choice for a unified approach to quantumness.

Here, the concept of quantumness of mutual information for a system of two spin-1/2 particles, named A and B, in the state described by the density matrix $\hat{\rho}^{\rm AB}$ is formulated by invoking explicitly the LHV theory [7]. To that end, the classical mutual information $I(\mathbf{a},\mathbf{b})$ of the spins is assumed to correspond to the joint probability $p(\epsilon_a^{\rm A};\epsilon_b^{\rm B})$ $(\epsilon_a^{\rm A},\epsilon_b^{\rm B}=\pm1)$ for the spin A to have the component $\epsilon_a^{\rm A}/2$ in the direction $\mathbf{a}$ and the spin B to have the component $\epsilon_b^{\rm B}/2$ in the direction $\mathbf{b}$, constructed by invoking the LHV theory [7]:

$$p(\epsilon_a^A;\epsilon_b^B)={\rm Tr}\left[\left(\frac{1}{2}+\epsilon_a^A\hat{S}_a^{\rm A}\right)\left(\frac{1}{2}+\epsilon_b^B\hat{S}_b^{\rm B}\right)\hat{\rho}^{AB}\right], \quad (1)$$

so that the classical mutual information is given by

$$I(\mathbf{a},\mathbf{b})=S(p(\epsilon_a^A))+S(p(\epsilon_b^B))-S(p(\epsilon_a^A,\epsilon_b^B)), \quad (2)$$

where $S(p(\{x_i\}_n))=-\sum_{\{x_i\}_n}p(\{x_i\}_n){\rm log}\ p(\{x_i\}_n)$ is Shannon entropy. On the other hand, quantum theoretic mutual information for a system described by $\hat{\rho}^{\rm AB}$ is

$$I_{\rm Q}(\hat{\rho}^{\rm AB})=S_{\rm Q}(\hat{\rho}^{\rm A})+S_{\rm Q}(\hat{\rho}^{\rm B})-S_{\rm Q}(\hat{\rho}^{\rm AB}), \quad (3)$$

where $S_{\rm Q}(\hat{\rho})=-{\rm Tr}[\hat{\rho}{\rm log}(\hat{\rho})]$ denotes the von Neumann entropy, and $\hat{\rho}^{\rm A}={\rm Tr}_{\rm B}(\hat{\rho}^{\rm AB})$, $\hat{\rho}^{\rm B}={\rm Tr}_{\rm A}(\hat{\rho}^{\rm AB})$ are the reduced density operators of the spins A and B respectively.

The LHV theoretic quantumness of information for the probabilities for the component of A in directions $\mathbf{a}$ and that of B in direction $\mathbf{b}$ may be characterized by $Q(\mathbf{a},\mathbf{b})$:

$$Q(\mathbf{a},\mathbf{b}) \;=\; I_{\rm Q}(\hat{\rho}^{\rm AB})-I(\mathbf{a},\mathbf{b}). \quad (4)$$

Different measures of quantumness are obtained by different choices of the directions $\mathbf{a}$ and $\mathbf{b}$. The choice of $\mathbf{a}$ and $\mathbf{b}$ may be made by recalling from [7] the considerations of the criterion for identifying quantumness in the correlations between observables. The said criterion in [7] is based on the properties of the joint quasiprobability in symmetric ordering for the eigenvalues of the components of each spin in three mutually orthogonal directions, one

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of which is the direction of the Bloch vector of that spin, and on the said quasiprobability for two of the three said components. That criterion identifies non-classical states of two or more spin-1/2 particles in agreement with the predictions based on other approaches, including the prediction of classicality of certain non-separable states. In view of that, **a** and **b** may be taken as the directions of the Bloch vectors $\langle\hat{\mathbf{S}}^{\rm A}\rangle$ and $\langle\hat{\mathbf{S}}^{\rm B}\rangle$ ($\langle\hat{P}\rangle = {\rm Tr}(\hat{P}\hat{\rho}^{\rm AB})$) of the spins A and B where $\hat{\mathbf{S}}^{\rm A}$ and $\hat{\mathbf{S}}^{\rm B}$ denote their respective spin vectors to define the LHV theoretic quantumness of mutual information as

$$Q_{\rm LHV} = I_{\rm Q}(\hat{\rho}^{\rm AB}) - I_{\rm LHV}, \tag{5}$$

$$I_{\rm LHV} \equiv I(\mathbf{a},\mathbf{b}), \quad \mathbf{a} = \frac{\langle\hat{\mathbf{S}}^{\rm A}\rangle}{|\langle\hat{\mathbf{S}}^{\rm A}\rangle|} \neq 0, \quad \mathbf{b} = \frac{\langle\hat{\mathbf{S}}^{\rm B}\rangle}{|\langle\hat{\mathbf{S}}^{\rm B}\rangle|} \neq 0. \tag{6}$$

The directions **a** and **b** are uniquely determined whenever $\langle\hat{\mathbf{S}}^{\rm A}\rangle \neq 0, \langle\hat{\mathbf{S}}^{\rm B}\rangle \neq 0$. The $Q_{\rm LHV}$ in this case is found to be identical with the measurement induced disturbance [5]. However, measurement induced disturbance does not specify **a** (**b**) uniquely when $\langle\hat{\mathbf{S}}^{\rm A}\rangle = 0$ ($\langle\hat{\mathbf{S}}^{\rm B}\rangle = 0$). The cases of vanishing average of one or both spin vectors are discussed in the following.

Let $\langle\hat{\mathbf{S}}^{\rm A}\rangle = 0$ but $\langle\hat{\mathbf{S}}^{\rm B}\rangle \neq 0$. In that case **a** in (6) can be any direction. The $I_{\rm LHV}$ in this case is defined to be the maximum of $I(\mathbf{a},\mathbf{b})$ over all **a**:

$$I_{\rm LHV} = {\rm max}_{\mathbf{a}} I(\mathbf{a},\mathbf{b}), \quad \langle\hat{\mathbf{S}}^{\rm A}\rangle = 0, \quad \mathbf{b} = \frac{\langle\hat{\mathbf{S}}^{\rm B}\rangle}{|\langle\hat{\mathbf{S}}^{\rm B}\rangle|} \neq 0. \tag{7}$$

The $Q_{\rm LHV}$ then turns out to be the same as the quantum discord for measurement on A if the eigenstates of $\hat{\mathbf{S}^{\rm B}} \cdot \mathbf{b}$ are also the eigenstates of the operator $\langle\pm,\mathbf{a}_m|\hat{\rho}^{\rm AB}|\pm,\mathbf{a}_m\rangle$ on B where $\mathbf{a}_m$ is the direction of optimization of spin A for evaluation of the quantum discord and $|\pm,\mathbf{a}_m\rangle$ are the eigenstates of $\hat{\mathbf{S}^{\rm A}} \cdot \mathbf{a}_m$.

If $\langle\hat{\mathbf{S}}^{\rm A}\rangle = \langle\hat{\mathbf{S}}^{\rm B}\rangle = 0$ then both, **a** and **b**, in (6) are non-unique. The $I_{\rm LHV}$ is then defined to be the maximum of $I(\mathbf{a},\mathbf{b})$ over all **a** and **b**:

$$I_{\rm LHV} = {\rm max}_{\mathbf{a},\mathbf{b}} I(\mathbf{a},\mathbf{b}), \quad \langle\hat{\mathbf{S}}^{\rm A}\rangle = \langle\hat{\mathbf{S}}^{\rm B}\rangle = 0. \tag{8}$$

The $I_{\rm LHV}$ in this case can be evaluated analytically exactly leadiing to the following exact expression for $Q_{\rm LHV}$:

$$Q_{\rm LHV} = I_{\rm Q}(\hat{\rho}^{\rm AB}) - \left[1 + H\left(\frac{1+{\rm C}}{2}, \frac{1-{\rm C}}{2}\right)\right]. \tag{9}$$

Here $H(x,y) = -x\log(x) - y\log(y)$, and $C = \max(|f_1|,|f_2|,|f_3|)$ where (f_1,f_2,f_3) are the eigenvalues of the matrix $\hat{C} \equiv \{C_{ij}\}$ where

$${\rm C}_{ij} = 4{\rm Tr}\left(\hat{S}_i^{\rm A}\hat{S}_j^{\rm B}\hat{\rho}^{AB}\right), \tag{10}$$

denotes correlation between the observables $\hat{S}_i^{\rm A}$ and $\hat{S}_j^{\rm B}$ of spins A and B with $\hat{S}_i^{\mu} = \hat{\mathbf{S}}^{\mu} \cdot \mathbf{e}_i$ ($\mu = {\rm A},{\rm B}$); ($i = 1,2,3$), and ($\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$) are the unit basis vectors in configuration space. The $Q_{\rm LHV}$ above is the same as the symmetric discord. Several states, like Werner state, and others for which analytic results for various discords are available, fall in the category of vanishing average directions of both the spins [6]. The $Q_{\rm LHV}$ for such states correspond to the special cases of (9).

As examples, consider first the pure state. The Bloch vector of each spin in this case is non-zero. By virtue of the considerations above, it follows that $Q_{\rm LHV}$ then is same as the measurement induced disturbance. It turns out also to be the same as the symmetric and the quantum discords.

As regards mixed states, recall that any mixed state of two qubits can be expressed as [8]

$$\begin{aligned}\hat{\rho}^{\rm AB} \;\; &= \;\; \frac{1}{4}I^{\rm A} \otimes I^{\rm B} + \sum_{i,j=1,2,3} w_{ij}\hat{S}_i^{\rm A} \otimes \hat{S}_j^{\rm B} \\ &\quad + \frac{r}{2}\hat{S}_3^{\rm A} \otimes I^{\rm B} + \frac{s}{2}I^{\rm A} \otimes \hat{S}_3^{\rm B}.\end{aligned} \tag{11}$$

The optimization involved in symmetric and other discords is generally a formidable task. We evaluate $Q_{\rm LHV}$ for some special cases for which analytic results for various discords are known.

In conclusion, though the concept of classicality of quantum states originates from the the local hidden variable theory, the existing measures of quantumness of information do not invoke that theory explicitly. Here, by invoking explicitly the local hidden variable theory, a measure of quantumness of mutual information for a system of two spin-1/2 particles is proposed. It circumvents the need of optimization when the Bloch vector of each spin is non-zero; the optimization is needed but can be performed analytically exactly when the Bloch vector of each spin vanishes, and is simplified when the Bloch vector of only one of the spins is zero.

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