Efficient quantum algorithm to construct arbitrary Dicke states

Kaushik Chakraborty[1] * Byung-Soo Choi[2] † Arpita Maitra[1] ‡ Subhamoy Maitra[1] §

[1] *Applied Statistics Unit, Indian Statistical Institute, 203 B T Road, Kolkata 700 108, India*
[2] *Department of Electrical and Computer Engineering, Duke University, Durham, NC 27708, USA*

Abstract. In this paper, we study efficient algorithms towards the construction of any arbitrary Dicke state. Our contribution is to use proper symmetric Boolean functions that involve manipulations with Krawtchouk polynomials. Deutsch-Jozsa algorithm, Grover algorithm and the parity measurement technique are stitched together to devise the complete algorithm. Further, motivated by the work of Childs et al (2002), we explore how one can plug the biased Hadamard transformation in our strategy. Our work improves the results of Childs et al (2002).

Keywords: Biased Hadamard Transform, Deutsch-Jozsa Algorithm, Dicke State, Grover Algorithm, Krawtchouk Polynomial, Parity Measurement, Symmetric Boolean Functions.

1 Introduction

Multipartite entanglement is one of the important areas in the field of quantum information that has many applications including quantum secret sharing. In this paper, we focus on the Dicke states [5], which are useful building blocks in realizing multipartite entanglement. The n-qubit weight w Dicke state, $|D_w^n\rangle$, is the equal superposition of all n-qubit states of weight w. We refer to [1, 8, 9, 10, 14, 15] and the references therein for detailed discussion.

After the invention of quantum information, many experimental setups have been proposed and tested to verify some theoretical properties. Most of experiments have been focused on the test of multipartite entanglement such as EPR, GHZ, and W states. Since the result of experimental tests depends on the steps for preparing, processing, and measuring, all steps should be refined as much as possible. Among them, the first priority is to prepare the target state with very high fidelity and with efficiency. In this work, therefore, we also focus on the efficient way to prepare certain multipartite quantum state.

In line of GHZ and W states, we have the Dicke state, $|D_w^n\rangle$, which an equal superposition state of all n-qubit states of weight w. Actually, Dicke state is more general state than GHZ and W states since W state is $|D_1^n\rangle$ and GHZ state is the superposition of $|D_0^n\rangle$ and $|D_n^n\rangle$. Therefore, the preparation method for Dicke state can be utilized for other general case as well. At the same time, similar to the above reason, Dicke state can be utilized for many applications such as secret sharing [15] and quantum networking [2]. Related to this, some previous works have been done that focussed on the experimental ways to prepare six-qubit Dicke state [15, 13] with fidelity 0.654 ± 0.024 and 0.56 ± 0.02, respectively.

While the main focus from the viewpoint of experimental physics is to actually provide the implementation of specific Dicke states, our focus is from theoretical algorithmic angle and the only result presented in this direction appeared in [1]. In this work, we show how one can efficiently construct Dicke states by using the combinatorial properties of symmetric Boolean functions, two well-known quantum algorithms, and the generalized parity measurement. By efficient, we mean that the resource requirements in terms of quantum circuits and number of execution steps is poly(n) to obtain $|D_w^n\rangle$.

Let us consider n-qubit states in the computational basis $\{0,1\}^n$ that can be written in the form $\sum_{x\in\{0,1\}^n} a_x|x\rangle$, where $\sum_{x\in\{0,1\}^n} |a_x|^2 = 1$. Thus, x can also be interpreted as a binary string and the number of 1's in the string is called the (Hamming) weight of x and denoted as $wt(x)$. Based on this an arbitrary Dicke state can be expressed as follows:

$$|D_w^n\rangle = \sum_{x\in\{0,1\}^n, wt(x)=w} \frac{1}{\sqrt{\binom{n}{w}}}|x\rangle.$$

Let us also define a symmetric n-qubit state as

$$|S^n\rangle = \sum_{x\in\{0,1\}^n} a_{wt(x)}|x\rangle, \text{ where } \sum_{i=0}^{n}\binom{n}{i}|a_i|^2 = 1.$$

2 Contribution

First, we show how one can prepare a symmetric n-qubit state with the property that $\binom{n}{w}|a_w|^2$ is $\Omega(\frac{1}{\sqrt{n}})$ by using Deutsch-Jozsa algorithm [4]. This requires certain novel combinatorial observations related to symmetric Boolean functions. Then the quantum state out of Deutsch-Jozsa algorithm is measured using the parity measurement technique [8] to obtain $|D_w^n\rangle$ with a probability $\Omega(\frac{1}{\sqrt{n}})$. Thus, $O(\sqrt{n})$ runs are sufficient to obtain the required Dicke state. Note that a direct approach to construct a symmetric state has been presented in [1] using biased Hadamard transform. While the order of probability to obtain Dicke state by ours and that of [1] are the same, enumeration results show that the exact probability values are better in our case than that of [1].

Further, motivated by the idea in [1], we improve our algorithm further with a modified Deutsch-Jozsa operator that involves the biased Hadamard transform. Since

*kaushik.chakraborty9@gmail.com
†bschoi3@gmail.com
‡arpita76b@rediffmail.com
§subho@isical.ac.in

biased Hadamard transform also helps to generate the target symmetric state, the overall probability to obtain the Dicke state increases.

Finally, we can also apply the Grover operator [7] before the measurement. Since Grover algorithm amplifies the amplitude of target symmetric state, this helps to reduce the necessary number of steps into $O(\sqrt[4]{n})$.

The brief non-technical description of the complete algorithm is as follows:

- We use combinatorial properties of Boolean functions, to be specific, symmetric Boolean functions and Krawtchouk polynomials [3, 6].
- We refer to the relationship [11] of Deutsch-Jozsa algorithm [4] with Walsh spectrum of Boolean functions to produce the symmetric states.
- Based on our combinatorial construction, we produce symmetric states with higher amplitudes for weight w states.
- We apply Grover's algorithm [7] to obtain quadratic improvement in the run time.
- We use parity measurement techniques [8] to obtain the Dicke state asked for.

3 Conclusion

In summary, we study several quantum algorithms to construct arbitrary Dicke state in a disciplined manner. The key idea is to find a suitable symmetric Boolean function for Deutsch-Jozsa algorithm for the given n and w, use of the Grover algorithm and the generalized parity measurement strategy. Further, we show that it is possible to obtain improved results using biased Hadamard transform suitably. Our results improve the probabilities obtained in [1] and provide faster method to construct Dicke states.

The problem open in this area is to characterize the enumeration results in case of modifying the Deutsch-Jozsa algorithm with biased Hadamard transform. Obtaining the exact bias in biased Hadamard transform with the corresponding symmetric function to optimize the probability corresponding to the Dicke state seems to be an interesting problem.

Though we look at the problem from theoretical angle, the algorithmic blocks used by us have experienced major advancement towards actual implementation. One may refer to [12, Section 7] for literature related to implementation of quantum gates as well as Deutsch-Jozsa algorithm, Grover algorithm and several measurement techniques. As example, the idea of implementing biased Hadamard transform is related to the Fabry-Perot cavity [12, Page 299].

References

[1] A. M. Childs, E. Farhi, J. Goldstone and S. Gutmann. Finding cliques by quantum adiabatic evolution. Quantum Information and Computation **2**, 181 (2002).

[2] A. Chiuri, C. Greganti, M. Paternostro, G. Vallone, and P. Mataloni. Experimental Quantum Networking Protocols via Four-Qubit Hyperentangled Dicke States. Physical Review Letters **109**, 173604 (2012).

[3] E. Demenkov, A. Kojevnikov, A. Kulikov and G. Yaroslavtsev. New upper bounds on the Boolean circuit complexity of symmetric functions. Information Processing Letters **110**, 264-267 (2010).

[4] D. Deutsch and R. Jozsa. Rapid solution of problems by quantum computation. Proceedings of Royal Society of London A **439**, 553-558 (1992).

[5] R. H. Dicke. Coherence in Spontaneous Radiation Processes. Physical Review **93**(1), 99 (1954).

[6] P. Feinsiver and R. Fitzgerald. The Spectrum of Symmetric Krawtchouk Matrices. *Linear Algebra & Applications* **235**, 121-139 (1996).

[7] L. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of 28th Annual Symposium on the Theory of Computing*, 212-219 (1996).

[8] R. Ionicioiu, A. E. Popescu, W. J. Munro and T. P. Spiller. Generalized parity measurements. Physical Review A **78**, 052326 (2008).

[9] N. Kiesel, C. Schmid, G. Toth, E. Solano, and H. Weinfurter. Experimental Observation of Four-Photon Entangled Dicke State with High Fidelity. Physical Review Letters **98**, 063604 (2007).

[10] I. E. Linington and N. V. Vitanov. Decoherence-free preparation of Dicke states of trapped ions by collective stimulated Raman adiabatic passage. Physical Review A **77**, 062327 (2008).

[11] S. Maitra and P. Mukhopadhyay. Deutsch-Jozsa Algorithm Revisited in the Domain of Cryptographically Significant Boolean Functions. In International Journal on Quantum Information **3**(2), 359-370 (2005).

[12] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2010.

[13] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger. Experimental Realization of Dicke States of up to Six Qubits for Multiparty Quantum Networking. Physical Review Letters **103**, 020503 (2009).

[14] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger. Experimental Realization of Dicke States of up to Six Qubits for Multiparty Quantum Networking. Physical Review Letters **103**, 020503 (2009).

[15] W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Toth, and H. Weinfurter. Experimental Entanglement of a Six-Photon Symmetric Dicke State. Physical Review Letters **103**, 020504 (2009).