How well can one jointly measure two incompatible observables on a given quantum state?

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Abstract. We consider the approximate joint measurement of two incompatible observables on a given quantum state, and present a tight relation characterizing the optimal trade-off between the error on one observable vs. the error on the other. As a particular case, our approach allows us to characterize the disturbance of an observable induced by the approximate measurement of another one; we introduce an even stronger error-disturbance relation for this scenario.

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1 Introduction

Uncertainty Relations, first suggested by Heisenberg in 1927 [1], are among the main pillars of quantum theory. One of the best known versions is for instance the one due to Robertson [2], which writes $\Delta A\, \Delta B \geq |C_{AB}|$, where ΔA, resp. ΔB, is the standard deviations of the measurement of an observable A, resp. B, on a quantum state $|\psi\rangle$, and $C_{AB} = \langle\psi|[A,B]|\psi\rangle/2i$.

It is often argued that such uncertainty relations imply that incompatible observables cannot be jointly measured on quantum states such that $C_{AB} \neq 0$, or that the measurement of one necessarily implies a disturbance on the other (which is in fact the original idea presented by Heisenberg in [1]). This is however not what the Robertson relation above tells, as it only bounds the statistical deviations of the measurement results of A and B, when either of each measurement is performed many times, on several independent copies of $|\psi\rangle$.

Instead of considering such statistical deviations on many measurements of *either* A or B, we address here the following problem: if A and B are incompatible and cannot be perfectly jointly measured on $|\psi\rangle$, it may still be possible to *jointly approximate* the measurement of both observables, at the price of introducing errors; what is then the optimal trade-off between the errors $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$ introduced in the measurement of A and B, respectively?

2 Approximate joint measurements

In order to approximate the measurement of an observable A on a quantum system in the state $|\psi\rangle$, a general strategy consists in measuring another, "approximate" observable $\mathcal{A}$, possibly on an extended Hilbert space—i.e., on the joint system composed of the state $|\psi\rangle$, and of an ancillary system in the state $|\xi\rangle$. In this picture, the impossible joint measurement of two incompatible observables A and B on $|\psi\rangle$ can thus be approximated by the perfect joint measurement of two compatible (i.e., commuting) observables $\mathcal{A}$ and $\mathcal{B}$ on $|\psi,\xi\rangle = |\psi\rangle \otimes |\xi\rangle$.

Following Ozawa (e.g. [3, 4]), we characterize the quality of the approximations $\mathcal{A}$ and $\mathcal{B}$ of A and B, respectively, by defining the *root-mean-square (rms) errors*

$$\epsilon_{\mathcal{A}} = \langle\psi,\xi|\, (\mathcal{A} - A \otimes \mathbb{1})^2 \,|\psi,\xi\rangle^{1/2}, \quad (1)$$

$$\epsilon_{\mathcal{B}} = \langle\psi,\xi|\, (\mathcal{B} - B \otimes \mathbb{1})^2 \,|\psi,\xi\rangle^{1/2}. \quad (2)$$

3 Error-trade-off relations for approximate joint measurements

The fact that quantum theory forbids perfect joint measurements of incompatible observables implies that the rms errors $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ can in general not take arbitrary values.

A common misconception is that Robertson's relation should still hold if the standard deviations ΔA and ΔB are simply replaced by the rms errors $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$, so that $\epsilon_{\mathcal{A}}\, \epsilon_{\mathcal{B}} \geq |C_{AB}|$. While this relation, often attributed to Heisenberg himself or to Arthurs and Kelly [5], can indeed be proven under some restrictive assumptions, it is worth emphasizing that in general it does *not* hold [6].

Only recently did Ozawa show [4] how one could derive a universally valid "uncertainty relation" for joint measurements, namely

$$\epsilon_{\mathcal{A}}\ \epsilon_{\mathcal{B}} + \Delta B\ \epsilon_{\mathcal{A}} + \Delta A\ \epsilon_{\mathcal{B}} \ \geq\ |C_{AB}|. \quad (3)$$

The three terms in Ozawa's relation come from three independent uses of Robertson's relation to different pairs of observables. While this indeed leads to a valid relation and allows one to exclude a large set of impossible values $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$, this is not optimal, as the three Robertson's relations (and therefore Ozawa's relation) in general cannot be saturated simultaneously.

3.1 A new, tight error-trade-off relation for joint measurements

Using a general geometric inequality for vectors in a Euclidean space, we could show [7] how to improve upon the sub-optimality of Ozawa's proof, and derive the following error-trade-off relation for approximate joint mea-

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surements:

$$\Delta B^2\, \epsilon_{\mathcal{A}}^2 + \Delta A^2\, \epsilon_{\mathcal{B}}^2 + 2\sqrt{\Delta A^2 \Delta B^2 - C_{AB}^2}\; \epsilon_{\mathcal{A}}\, \epsilon_{\mathcal{B}} \;\geq\; C_{AB}^2. \tag{4}$$

It can easily be checked that Ozawa's relation (3) can directly be derived from our new relation above. Furthermore, not only is our relation stronger than Ozawa's, it is actually *tight*: for any A, B and $|\psi\rangle$, any values $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ saturating inequality (4) can be obtained [7]. Hence, contrary to previously derived relations, our new one does not only tell what *cannot* be done quantum mechanically, but also what *can* be done.

3.2 The error-disturbance scenario and the same-spectrum assumption

The error-disturbance scenario, as first discussed by Heisenberg [1], can be treated as a particular case of the general framework for approximate joint measurements.

In this context, one considers the disturbance $\eta_{\mathcal{B}}$ in the statistics of one observable, B, due to the unsharp measurement of another observable, A. The approximation of A can again be described by the measurement of an observable $\mathcal{A}$, while the subsequent measurement of B on the disturbed system can be written as the measurement of an observable $\mathcal{B}$ on the original state. Using the same formalism as in the joint measurement framework, the rms error $\epsilon_{\mathcal{B}}$ is now interpreted as the *rms disturbance* $\eta_{\mathcal{B}}$ of B, with formally the same definition [3]: $\eta_{\mathcal{B}} = \epsilon_{\mathcal{B}}$ as defined in (2).

Any error-trade-off relation derived in the more general framework of joint measurements thus remains valid in this error-disturbance scenario. In particular, when interpreting $\epsilon_{\mathcal{B}}$ as the rms disturbance $\eta_{\mathcal{B}}$, Ozawa's relation (3) writes

$$\epsilon_{\mathcal{A}}\; \eta_{\mathcal{B}} + \Delta B\; \epsilon_{\mathcal{A}} + \Delta A\; \eta_{\mathcal{B}} \quad\geq\quad |C_{AB}|. \tag{5}$$

This error-disturbance relation was actually introduced by Ozawa before its previous version (3) for joint measurements [3]. In a similar manner, our new error-trade-off relation (4) also implies a new error-disturbance relation, by simply replacing $\epsilon_{\mathcal{B}}$ by $\eta_{\mathcal{B}}$.

The difference with the previous, more general scenario of joint measurements is however not merely in the interpretation of $\epsilon_{\mathcal{B}}$. A crucial point is that as the approximate measurement $\mathcal{B}$ corresponds to the actual measurement of B on the disturbed system, it necessarily has the same spectrum as B—which was not assumed previously. Because of this constraint, one may expect stronger restrictions on the possible values of $\eta_{\mathcal{B}}$ to hold, and that stronger "error-disturbance relations" can be derived.

To illustrate this, we could show [7] that for the case of a dichotomic observable B with eigenvalues ± 1 (such that $B^2 = \mathbb{1}$), and for a state $|\psi\rangle$ for which $\langle B\rangle = 0$ (which implies $\Delta B = 1$), if one imposes the same spectrum assumption to $\mathcal{B}$ (i.e. that $\mathcal{B}^2 = \mathbb{1}$ as well), then an analogous relation to (4) holds, where $\epsilon_{\mathcal{B}}$ is replaced by $\eta_{\mathcal{B}}\sqrt{1 - \frac{\eta_{\mathcal{B}}^2}{4}}$. If A is also such that $A^2 = \mathbb{1}$ and $\langle A\rangle = 0$, and if one also imposes the same-spectrum assumption to $\mathcal{A}$, then one can derive the error-disturbance [7]

$$\epsilon_{\mathcal{A}}^2\Big(1 - \frac{\epsilon_{\mathcal{A}}^2}{4}\Big) + \eta_{\mathcal{B}}^2\Big(1 - \frac{\eta_{\mathcal{B}}^2}{4}\Big) + 2\sqrt{1 - C_{AB}^2}\; \epsilon_{\mathcal{A}}\sqrt{1 - \frac{\epsilon_{\mathcal{A}}^2}{4}}\; \eta_{\mathcal{B}}\sqrt{1 - \frac{\eta_{\mathcal{B}}^2}{4}} \;\geq\; C_{AB}^2. \tag{6}$$

This new error-disturbance relation is strictly stronger than (4) (and than Ozawa's relation (5)). Furthermore, we could show [7] that it is tight when $|\langle\psi|AB|\psi\rangle| = 1$: for any A, B and $|\psi\rangle$ satisfying the constraints above, one can reach any values $(\epsilon_{\mathcal{A}}, \eta_{\mathcal{B}})$ that saturate the inequality, using approximate measurements such that $\mathcal{A}^2 = \mathcal{B}^2 = \mathbb{1}$.

4 Conclusion

Our new error-trade-off relation (4) quantifies precisely the optimal trade-off between the rms errors $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$ introduced in any approximate joint measurement of A and B, thus answering the question posed in our title. In the case where one imposes that the approximations should have the same spectrum as A and B, one can derive stronger constraints, as e.g. our error-disturbance relation (6) for the case of ± 1-valued observables.

The *tightness* of these relations is a crucial feature: they do not only indicate what *cannot* be done quantum mechanically, but also what *can* be done. Two recent experiments [8, 9] verified the validity of Ozawa's error-disturbance relation (5) (and the violation of the "Heisenberg-Arthurs-Kelly relation" $\epsilon_{\mathcal{A}}\, \eta_{\mathcal{B}} \geq |C_{AB}|$), but were nowhere near saturating it—which is indeed in general not possible. However, an adequate setup (such as the one of [9]) should allow one to saturate our error-disturbance relation (6), as well as our error-trade-off relation (4) if the approximate measurements are not restricted to output eigenvalues of A and B.

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