Non-classicality and entanglement of symmetric multiqubit states

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Abstract. Quantum vs classical conceptions of nature continue to draw attention from the early days of quantum theory. Understanding decisive quantum features for which no classical explanation exist and their interrelations is of foundational significance. Further, identifying non-classical features is important in information processing tasks as it offers insights as to why quantum protocols outperform their classical counterparts. Here we focus on two illustrious notions of non-classicality: negativity of P phase-space representation and entanglement in symmetric multiqubit systems. We prove that in permutation symmetric multiqubit states and show that non-classicality implies entanglement and vice versa.

Keywords: non-classicality, entanglement, spin coherent states, P-representation

1 Introduction

Non-classical quantum states draw attention both from conceptual point of view and as a resource for quantum information technology. Tests to explore whether a quantum system exhibits *non-classicality* has gained renewed attention. These investigations provide insights into how quantum description of nature turns out be inevitable. Widely accepted notion of non-classicality emerged from quasi-probability distributions of quantum systems [1]. Failure of the Sudarshan-Glauber P-function [2, 3] from being a true classical probability density offers a decisive signature of *non-classicality* in single mode quantum radiation. On the other hand, information theoretic concepts define *non-classical correlations* in a composite quantum system [5, 6, 7]. A comparision of these two celebrated approaches in bipartite bosonic states revealed that *the notion of nonclassicality stemming from P representation is inequivalent to that emerging from information-theoretic arguments* [8]. In this paper, we investigate the same question in multiqubit symmetric systems and we find that the notion of non-classicality emerging from the P representation of spins [9] is a necessary and sufficient condition for entanglement [4].

2 Sudarshan-Glauber diagonal coherent state representation or P representation

In 1963, Sudarshan and Glauber introduced the celebrated diagonal coherent state representation or the P representation [2, 3] to characterize states of quantized radiation fields in terms of a classical phase-space like description. In this representation the density matrix $\hat{\rho}$ is expressed in terms of a classical function $P(\alpha)$

$$\hat{\rho} = \int \mathrm{d}^2\alpha\, P(\alpha)\, |\alpha\rangle\langle\alpha|. \qquad (1)$$

where $|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, are coherent states of light; $\hat{D}(\alpha) = e^{\hat{a}^\dagger\alpha-\hat{a}\alpha}$ denotes the displacement operator and $|0\rangle$ is the vacuum state of the radiation field.

Sudarshan-Glauber P representation of an arbitrary quantum state *formally* resembles a statistical mixture of coherent states of radiation. For a single mode coherent state, the weight function of the P representation reduces to a δ function in the phase space. However, for a large class of quantum states the P function cannot be interpreted as classical probability density as it can assume negative values or is more singular than delta function. Consequently, states with ill behaved P-function are referred to as non-classical.

3 Spin coherent state representation and non-classicality of symmetric multiqubit states

P representation has been extended to discrete spin states too. Arecchi et. al. [9] introduced an analog of diagonal spin coherent state representation:

$$\hat{\rho} = \int \mathrm{d}\Omega\, P(\theta,\phi)\, |\theta,\phi\rangle\,\langle\theta,\phi|, \qquad (2)$$

where $\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\,\mathrm{d}\phi$; $|\theta,\phi\rangle$ denote the spin coherent states

$$|\theta,\phi\rangle = \exp(\tau\hat{S}_+ - \tau^*\,S_-)\,|S,-S\rangle,\ \tau = \frac{1}{2}\,\theta\, e^{-i\phi}. \qquad (3)$$

Here $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ are ladder spin operators; $\{|S,\,M\rangle, -S \leq M \leq S\}$ denote simultaneous eigen states

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of the squared spin operator $\hat{S}^2$ and z component of spin $\hat{S}_z$.

Classicality of spin states, based on well behaved P representation has been explored by Giraud et. al. [10, 11]. Luis and Rivas [12] showed that spin squeezing [13] manifestly reflects the failure of P function to be a classical probability density. They also investigated simple operational procedures revealing the violation of classical statistical bounds by non-classical spin states. In addition to studying the non-classicality of single spin systems, Giraud et. al. [10, 11] analyzed the implications of P representation on entanglement in bipartite spin systems. They showed that a bipartite system consisting of two qubits is separable if and only if their P function is positive. However, in the case of a coupled system consisting of spin-1/2 and spin-1, it was recognized that separable states too can exhibit non-classicality manifested in terms of their non-positive or singular P function.

Here, we focus on symmetric N qubit states [14] for our investigation. The P-representation of symmetric N-qubit states has the form,

$$\begin{aligned} \hat{\rho}_{\rm sym}(N) &= \int {\rm d}\Omega\, P(\theta,\phi)\, |\theta,\phi\rangle\, \langle\theta,\phi| \\ &= \int {\rm d}\Omega\, P(\theta,\phi)\, \hat{\rho}_{(\theta,\phi)} \otimes \hat{\rho}_{(\theta,\phi)} \ldots \otimes \hat{\rho}_{(\theta,\phi)} \end{aligned} \tag{4}$$

where $\hat{\rho}_{(\theta,\phi)} = |1_{(\theta,\phi)}\rangle\langle 1_{(\theta,\phi)}|$; $|1_{(\theta,\phi)}\rangle$ denotes qubit spin-down state in the direction (θ,ϕ). Whenever $P(\theta,\phi)$ is positive, the symmetric multiqubit state $\hat{\rho}_{\rm sym}$ admits a separable decomposition. Conversely, we also prove that the set of *all* separable symmetric multiqubit states

$$\hat{\rho}^{(\rm sep)}_{\rm sym}(N) = \sum_w p_w\, \hat{\rho}_{(\theta_w,\phi_w)} \otimes \hat{\rho}_{(\theta_w,\phi_w)} \ldots \otimes \hat{\rho}_{(\theta_w,\phi_w)} \tag{5}$$

(where $0 \leq p_w \leq 1$; $\sum_w p_w = 1$) necessarily admit a positive P representation [4] i.e., the P function of the set of *all* separable symmetric multiqubit states is a well behaved convex sum:

$$P(\theta,\phi) = \sum_w p_w\, \delta(\theta_w - \theta)\, \delta(\phi_w - \phi). \tag{6}$$

Entangled symmetric multiqubit states cannot be decomposed as mixtures of coherent states with positive weights as in (5), and are *non-classical.* More specifically, entanglement manifests itself in terms of non-positive P function in symmetric multiqubit states.

4 Conclusions

Two celebrated notions of non-classicality are (i) the failure of P phase space representation from being a legitimate probability density function and (ii) quantumness of correlations in composite systems. A comparison of these two approaches in bosonic systems revealed maximal inequivalence [8]. This motivated our investigation in symmetric multiqubit systems. We prove here that the notion of non-classicality arising from a ill behaved P representation and entanglement imply each other in symmetric multiqubit systems.

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