Universal implementation of energy eigenbasis measurement

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Abstract. We show a scheme to universally implement a projective measurement in the energy eigenbasis on a system evolving by an unknown Hamiltonian based on the phase estimation algorithm. To apply the phase estimation algorithms for unknown Hamiltonian systems, two new algorithms are introduced. One is for asymptotically but universally implementing a controlled-unitary operation of a unitary operation of Hamiltonian evolution. Another is a new deterministic quantum computation with one pure qubit (DQC1) algorithm for evaluating the absolute value of the unitary operation.

Keywords: quantum measurement, dynamical decoupling, phase estimation

1 Motivation

The postulates of quantum measurement assert that any Hermitian operator is associated to an observable and its projective measurement is possible. On the other hand, another postulate dictates that systems evolve according to some Hamiltonian. Thus the projective measurement of energy should be possible on a system evolving according to its Hamiltonian. Implementation of the energy eigenbasis projective measurement requires manipulation from the outside of the system evolving by the Hamiltonian. This is possible if there is another system on which we can implement any quantum map at our will coupling to the system. A system with such high controllability is a quantum computer. We assume that the quantum computer can operate in a time scale much faster then the Hamiltonian dynamics of the system.

To perform an energy eigenbasis measurement on a system evolving by an *unknown Hamiltonian*, we can use the following brute-force method. First, we estimate the Hamiltonian by process tomography [1] to find a description of the Hamiltonian. Then we compute the eigenvectors of the Hamiltonian and perform a unitary operation on the system that maps the energy eigenbasis to the computational basis. We perform a projective measurement in the computational basis. The time required to perform this method depends on the dimension of the system that is exponential in terms of the system size.

The *phase estimation algorithm* [2] provides the better implementation of the energy eigenbasis measurement [3]. Its running time does not depend on the dimension of the system but on the required accuracy. This algorithm works if a controlled version of Hamiltonian dynamics is provided. One proposal to obtain the controlled version of the dynamics from the Hamiltonian dynamics [4] assumes that the input is encoded in a particular subspace and that there is another subspace on which the Hamiltonian acts as the identity operator. These assumptions are satisfied in particular setups such as in linear optical quantum computation using photon qubits, but cannot be generally applied to other settings.

In this talk, we propose an *efficient* and *universal* algorithm that implements the controlled version of Hamiltonian dynamics to perform the phase estimation algorithm for general systems evolving by unknown Hamiltonian. This algorithm works approximately. We also propose a new *DQC1 algorithm* that evaluates the efficiency of the approximation.

2 Phase estimation algorithm

The phase estimation algorithm is a quantum algorithm to estimate the phase factor $0 \leq \theta_i < 2\pi$ of the eigenvalue $e^{i\theta_i}$ of a unitary operation U, when an eigenstate $|\theta_i\rangle$ is given as an input state. The algorithm uses controlled-unitary operations of $U, U^2, U^{2^2}, \cdots, U^{2^N}$ where N denotes the number of control qubits. A controlled-unitary operation C_U of an unitary operation U is defined by

$$C_U := |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes U \tag{1}$$

on $\mathcal{H}_c \otimes \mathcal{H}_t$ where the Hilbert spaces of the control system and the target system are represented by $\mathcal{H}_c = \mathbb{C}^2$ and $\mathcal{H}_t = \mathbb{C}^d$, respectively. As a map on density matrices, we denote this unitary evolution by $\mathcal{C}_{U(t/m)}$.

As N increases, the probability to obtain an outcome outside a fixed range of an eigenvalue decreases exponentially in terms of N, whereas the total calling time of U increases exponentially. If we apply the phase estimation algorithm to an arbitrarily superposed input state $|\phi\rangle = \sum_i \alpha_i |\theta_i\rangle \in \mathcal{H}_t$ where $\sum_i |\alpha_i|^2 = 1$, the algorithm implements a projective measurement $\{|\theta_i\rangle\langle\theta_i|\}$ on $\mathcal{H}_t$ as $N \to \infty$.

3 Universal controllization

We propose an algorithm that *asymptotically* and *universally* implements a controlled unitary operation when the unitary operation is given as $U(t) = e^{-iHt}$ for an *unknown* Hamiltonian H.

The algorithm is the following. We add an ancilla system where its Hilbert space is represented by $\mathcal{H}_a = \mathbb{C}^d$ and its initial state is prepared in a maximally mixed state $\mathbb{I}/d$. We divide the time evolution $U(t)$ on $\mathcal{H}_t$ into m iterations of $U(\frac{t}{m}) = e^{-iH\frac{t}{m}}$, and insert Fredkin gates

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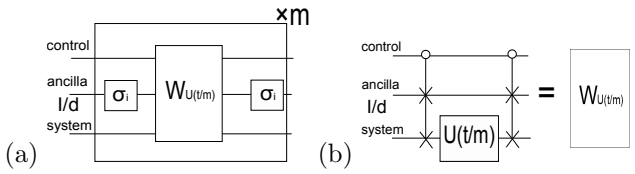

Figure 1: (a) A quantum circuit representation of the algorithm approximately implementing $C_{U(t)}$. The two generalized Pauli operations σ_i in a sequence are identical, but they have to be chosen randomly for each iteration. (b) A quantum circuit representation of the pseudo controllization gate $W_{U(t/m)}$.

and randomly chosen general Pauli operations before and after each $U(\frac{t}{m})$ as shown in Fig. 1. Due to the dynamical decoupling theorem [5], the effects of the two random Pauli operations in each iteration lead the ancilla state to $\mathbb{I}/d$ and the ancilla system is decoupled from the other systems. Thus m iterations of this procedure implement a map $\Gamma^m_{U(t/m)}$ that is an approximation of $\mathcal{C}_{e^{i\varphi}}\mathcal{C}_{U(t)}$, where $\mathcal{C}_{e^{i\varphi}} := \mathcal{C}_{e^{i\varphi}\mathbb{I}}$ is originated in the global phase of $U(t/m)$ and is inevitably added in the controllization process. The effect of this additional $\mathcal{C}_{e^{i\varphi}}$ can be ignored in the energy eigenbasis measurement since it results in the base-point shift of the energy eigenvalues.

The difference between the maps $\mathcal{C}_{e^{i\varphi}}\mathcal{C}_{U(t)}$ and $\Gamma^m_{U(t/m)}$ can be evaluated in terms of the diamond norm by

$$\left|\left|\mathcal{C}_{e^{i\varphi}}\mathcal{C}_{U(t)} - \Gamma^m_{U(t/m)}\right|\right|_\diamond = 1 - (a_{U(t/m)})^m \qquad (2)$$

where

$$a_{U(t/m)} := |\mathrm{Tr}[U(t/m)]/d|. \qquad (3)$$

We call $(a_{U(t/m)})^m$ as a coherence factor. The coherence factor can be evaluated by

$$\begin{aligned}(a_{U(t/m)})^m &= O[((\mathrm{Tr}H)^2 - \mathrm{Tr}H^2)t^2/(d^2m)] \\ &\leq 1 + O[(\Delta_{\max})^2t^2/m], \end{aligned} \qquad (4)$$

where $E_i = -\theta_i m/t$ is an eigenvalue of H corresponding to the eigenstate $|\theta_i\rangle$ and $\Delta_{\max}$ is the maximum energy difference (the largest eigenvalue minus the smallest eigenvalue) of H. Thus the right hand side of Eq. (2) can be bounded by $[(\Delta_{\max})^2t^2/m]$.

4 Algorithm evaluating coherence factor

The coherence factor $(a_{U(t/m)})^m$ can be evaluated in the following manner. Consider the following random unitary operation,

$$\mathcal{V}'_{U(t/m)}(\rho) := \frac{1}{D^2}\sum_{i,j} V^{(i,j)}_{U(t/m)}\rho(V^{(i,j)}_{U(t/m)})^\dagger, \qquad (5)$$

where

$$V^{(i,j)}_{U(t/m)} = (\mathbb{I}\otimes\sigma_i\otimes\sigma_j)W_{U(t/m)}(\mathbb{I}\otimes\sigma_i\otimes\sigma_j). \qquad (6)$$

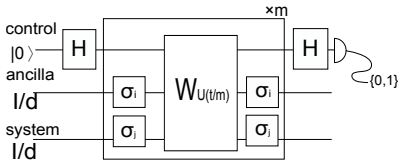

Figure 2: Algorithm for evaluating factor $(a_{U(t/m)})^m$. We perform dynamical decoupling which separates the control system from the target and the ancilla system.

The algorithm depicted in Fig. 2 applies $\mathcal{V}'_{U(t/m)}$ for m times. By the dynamical decoupling effect similarly used in the controllization algorithm, the resulting state after the Hadamard gate is given by $\tilde{\rho}\otimes\mathbb{I}/d\otimes\mathbb{I}/d$, where

$$\tilde{\rho} = \frac{\mathbb{I} + a^{2m}_{U(t/m)}\sigma_z}{2}. \qquad (7)$$

The coherence factor can be calculated from the probability of obtaining 0 or 1 in the last measurement according to $\mathrm{Tr}[\tilde{\rho}\sigma_z] = (a_{U(t/m)})^{2m}$.

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