A minimal state-dependent proof of measurement contextuality for a qubit

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Abstract. We show that three unsharp two-valued qubit measurements are enough to violate a noncontextual inequality, referred to as Specker's inequality, in a state-dependent manner. We obtain the optimal state-dependent quantum violation of this inequality allowed by qubit POVMs. Besides, we show that qubit POVMs do not allow a state-independent violation of Specker's inequality. We thus provide a minimal state-dependent proof of measurement contextuality requiring one qubit and three unsharp measurements. Our result establishes a novel no-go theorem for generalized-noncontextual models of these measurements.

Keywords: Contextuality, Joint measurability, Quantum Foundations

1 Introduction

The doubts of EPR [1] and the theorems of Bell, Kochen and Specker [2, 3] established the impossibility of some very natural hidden variable models of quantum theory, characterized by locality and noncontextuality. Spekkens recently generalized this class of models to generalized-noncontextual ontological models, defining them in a manner that is applicable to any operational theory rather than quantum theory alone [4]. The central motivation of these models is to illuminate the distinction between the classical and the quantum world in a manner that is mathematically sound and, hopefully, experimentally testable, e.g., through violations of Bell inequalities [5] and, more recently, experiments demonstrating contextuality for indivisible systems [6, 7, 8].

We consider a noncontextual inequality obtained by Liang et al. [10] based on a version of contextuality that was first discussed by Specker in 1960 [11], well before the Bell-Kochen-Specker theorem. This inequality concerns the stength of anti-correlations in the simplest conceivable contextuality scenario which we refer to as Specker's scenario. We also refer to the inequality as Specker's inequality. A contextuality scenario is a collection of subsets, called 'contexts', of the set of all measurements being considered. A context refers to measurements that can be jointly implemented. Specker's scenario requires three two-valued measurements, $\{M_1, M_2, M_3\}$, to allow for three non-trivial contexts: $\{\{M_1, M_2\}, \{M_1, M_3\}, \{M_2, M_3\}\}$. Each measurement takes values in $\{+1, -1\}$. On assigning outcomes $\{+1, -1\}$ noncontextually to the three measurements $\{M_1, M_2, M_3\}$, it becomes obvious that the maximum number of anti-correlated contexts possible in a single assignment is two, e.g., for the assignment $\{M_1 \rightarrow +1, M_2 \rightarrow -1, M_3 \rightarrow +1\}$, $\{M_1, M_2\}$ and $\{M_2, M_3\}$ are anti-correlated but $\{M_1, M_3\}$ is not. This puts an upper bound of $\frac{2}{3}$ on the probability of anti-correlation when a context is chosen uniformly at random. Specker's scenario precludes projective measurements because a set of three pairwise commuting projective measurements is trivially jointly measurable and cannot show contextuality. One may surmise that it represents a kind of contextuality that is not seen in quantum theory. However, as Liang et al. [10] showed, this contextuality scenario can be realized using noisy spin-1/2 observables. They showed that if one does not assume outcome determinism for unsharp measurements and models them stochastically but noncontextually, then this generalized-noncontextual model [4] for noisy spin-1/2 observables will obey a bound of $1 - \frac{\eta}{3}$, where $\eta \in [0, 1]$ is the sharpness associated with each observable. Formally,

$$R_3 \equiv \frac{1}{3} \sum_{(ij) \in \{(12),(23),(13)\}} \Pr(M_i \neq M_j) \leq 1 - \frac{\eta}{3}, \quad (1)$$

where $\Pr(M_i \neq M_j)$ is the probability of anti-correlation between measurements M_i and M_j. Measurement statistics that always shows perfect anti-correlation between any two measurements sharing a context is maximally contextual, i.e., $R_3 = 1$. After giving examples of orthogonal and trine spin-axes that did not seem to show a violation of this inequality, Liang et al. [10] left open the question of whether such a violation exists. They conjectured that all such triples of POVMs will admit a generalized-noncontextual model [4], i.e., Specker's inequality (1) will not be violated.

We show that this is not the case [9]. In particular, we deal with triples of unsharp qubit POVMs in full generality, rather than only considering special cases like orthogonal and trine spin axes, and show that while they do not admit a state-independent violation of Specker's inequality, they do allow a state-dependent violation. We also obtain the optimal state-dependent violation of this inequality allowed by qubit POVMs.

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2 Results

The three POVMs considered, $M_k = \{E^k_+, E^k_-\}$, $k \in \{1,2,3\}$, are noisy spin-$\frac{1}{2}$ observables of the form

$$E^k_\pm \equiv \frac{1}{2}I \pm \frac{\eta}{2}\vec{\sigma}.\hat{n}_k, \quad 0 \leq \eta \leq 1, \tag{2}$$

for three measurement directions $\{\hat{n}_k\}$ and sharpness parameter η. Our main results are:

Theorem 1 *There exists no state-independent violation of Specker's inequality, $R_3 \leq 1-\frac{\eta}{3}$, using a triple of qubit POVMs, $\{M_k\} = \{E^k_\pm\}_{k\in\{1,2,3\}}$, that are pairwise jointly measurable but not triplewise jointly measurable.*

A state-independent violation would mean that contextuality can be demonstrated by implementing the pairwise qubit measurements on *any arbitrary* qubit state. Theorem 1 rules out such a violation of Specker's inequality for a qubit.

Theorem 2 *The optimal violation of Specker's inequality corresponds to coplanar measurements along $\{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$ such that $\hat{n}_1.\hat{n}_2 = \hat{n}_1.\hat{n}_3 \to -1$, $\hat{n}_2.\hat{n}_3 = 2(\hat{n}_1.\hat{n}_2)^2 - 1$, and $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ if the plane of measurements is the ZX plane. Under this choice of state and measurement directions, the lower and upper bounds on η are given by $\eta_l \to \frac{2}{3} \approx 0.6667$ and $\eta_u \to 1$, and the optimal violation of Specker's inequality approaches (but is strictly less than) $\frac{2}{27} \approx 0.0741$ or 7.41% for $\eta \to \eta_l$.*

The quantum probability of anti-correlation for this optimal violation approaches (but is strictly less than) $\frac{23}{27} \approx 0.8519$ for qubit measurements. This state-dependent violation occurs when the three qubit POVMs are chosen to lie in a plane of the Bloch sphere passing through its centre and the qubit is prepared in a pure state along the axis perpendicular to this plane. Thus we need to prepare the qubit in a special state to allow violation of Specker's inequality—one cannot obtain a violation for any arbitrary state given a fixed set of three qubit POVMs. We refer the reader to [9] for details of the proof, in particular the construction of pairwise joint POVMs for the three unsharp measurements.

3 Discussion

Contextuality arises from the non-existence of a global joint probability distribution over measurement outcomes that can reproduce the measurement statistics predicted by quantum theory. Traditionally, contextuality has been shown with respect to noncontextual hidden variable models of projective measurements for Hilbert spaces of dimension three or greater [3, 12, 13]. While a state-independent proof of contextuality holds for any arbitrary state, a state-dependent proof requires a special choice of the prepared state depending on the measurements. The minimal state-independent proof of traditional contextuality requires a qutrit and 13 projectors [13, 14]. The minimal state-dependent proof [12, 15], first given by Klyachko et al. [12], requires a qutrit and five projectors. What we have shown in Ref. [9] is that a simpler contextuality scenario, viz. Specker's, is realizable in quantum theory if one moves beyond projective measurements and considers the possibilities allowed by qubit POVMs. This allows us to give a state-dependent proof of generalized contextuality using a qubit and three unsharp measurements. Our proof is minimal in this sense.

Besides, our result hints at the fact that perhaps all contextuality scenarios may be realizable and contextuality for these may be demonstrated in quantum theory if we consider the possibilities that general quantum measurements allow. In particular, scenarios that involve pairwise compatibility between all measurements but no global compatibility may be realizable within quantum theory. Specker's scenario is the simplest such example we have considered.

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