Multipartite entanglement quantification in weighted graph states

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Abstract. We consider the question of evaluating multipartite entanglement measures for weighted graph states. Weighted graph states are a natural generalisation of the usual graph states used in quantum information processing. First we present two different methods of evaluating three multipartite entanglement measures in graph states, namely the Schmidt measure, the relative entropy of entanglement and the geometric measure. One method relies on stabiliser formalism while the other is inspired by PEPS construction. We then focus on the second method and study whether it can be generalised to the case of weighted graph states.

Keywords: multipartite entanglement measures, weighted graph states.

1 Introduction

Quantification of entanglement in multipartite states is one of the fundamental problems in quantum information theory. We concentrate on three measures of multipartite entanglement: Schmidt measure, relative entropy of entanglement and geometric measure. As all three measures are defined as minimisations of distances in Hilbert space or over all linear decompositions into product states they are extremely hard to compute analytically.

One particular class of multipartite quantum states whose entanglement properties can be studied analytically to a high degree are the graph states [1]. They play an important role as resource states in measurement-based quantum computation and in some communication protocols such as quantum secret sharing. The entanglement properties have been studied in a number of settings. A unified picture of the three entanglement measures in graph states has been presented in [2].

Graph states offer an idealised description of real physical systems interacting via an Ising-type interaction. To go beyond this simplification we have to consider weighted graph states [3]. Now the pairwise interaction differs for various pairs of qubits. This makes them a natural model for strongly coupled spin chains interacting via a long-range Ising-type interaction. So far weighted graph states have found numerous uses in describing various disordered systems such as spin gases [3] as well implementations for a number of quantum information processing tasks such as producing random circuits. Another motivation to study weighted graph states arises from experimental considerations of creating graph states.

The entanglement properties of these interesting states has been studied in [3]. But so far these studies were limited to the case of bipartite entanglement. In this work we address the question whether multipartite measures of entanglement can be easily evaluated for these states and highlight some of the difficulties encountered in doing so.

To achieve this goal we first outline how the three different multipartite entanglement measures can be evaluated for a large class of pure graph states by solving a single problem from graph theory, namely the maximum independent vertex set problem. Doing this allows us to construct the minimal linear decomposition of the graph state into product states as well as the closest separable state (CSS) and the closest product states (CPS).

We then focus on presenting a new method of constructing the CSS using an approach inspired by projected entangled pairs description of quantum states and demonstrate how this method can be adapted to find an upper bound on the relative entropy of entanglement of weighted graph states.

2 Results

Consider a pure N-qubit state $|\psi\rangle$ and assume that it can be written in the generalized Schmidt decomposition as $|\psi\rangle = \sum_i \xi_i |\psi_i^1\rangle \otimes \ldots \otimes |\psi_i^N\rangle$. The *Schmidt measure* is defined as $E_S(|\psi\rangle) := \log R_{min}$, where R_{min} is the minimal number of terms in the expansion of $|\psi\rangle$ over all linear decompositions into product states. The *relative entropy od entanglement* is defined as $E_R(\rho) := \min_{\omega \in SEP} S(\rho||\omega)$, where ρ in our case is a rank-one projector onto $|\psi\rangle$, S is the quantum realtive entropy and the minimization is carried over all mixed separable states. The *geometric measure* is defined as $E_G(|\psi\rangle) := \min_{|\phi\rangle \in PROD} -\log |\langle\phi|\psi\rangle|^2$, where the minimization is carried over all pure product states.

Graph is a pair $G = (V, E)$, where V is the vertex set and $E \subseteq [V]^2$ is the set of edges connecting the vertices. The *neighborhood* of vertex $a \in V$, denoted by N_a, is the set of all vertices adjacent to vertex a, $N_a := \{b \in V | (a, b) \in E\}$. A *vertex coloring* is a map $c : V \to S$, such that $c(v) \neq c(w)$ when vertices v and w are adjacent. S is the set of colors needed to color the graph. A graph is called *bipartite* if it is two-colorable. An *independent set* is a set of vertices that are pairwise non-adjacent. The largest such set is the *maximum independent set*, denoted by $\alpha(G)$. The complement set is called the *minimum vertex cover*, $\beta(G)$. *Matching* $M(G)$

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is a set of independent edges, that is edges that are not incident on the same vertex. Largest such set is called *maximum matching*, denoted by $M_{max}(G)$. Vertices that are part of the matching are referred to as *matched*. If the entire vertex set is matched then the matching is *perfect*.

Graph state $|G\rangle$ is the unique, common $+1$ eigenvector in $(\mathcal{C}^2)^V$ to N independent commuting observables $g_i := X_i \bigotimes_{j\in N_i} Z_j$. The Abelian group generated by $\{g_i\}_{i=1}^N$ is the stabilizer $\mathcal{S}$. $|G\rangle$ can be prepared by placing a qubit in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ at each vertex and applying two-qubit entangling control-Z gate $CZ = \mathrm{diag}\{1,1,1,-1\}$ between adjacent vertices. *Weighted graph state* $|WG\rangle$ can be prepared in a similar fashion but instead of a control-Z gate one needs to apply a control-phase gate $CR_{\phi_{ij}} = \mathrm{diag}\{1,1,1,e^{i\phi_{ij}}\}$, where the phase ϕ_{ij} depends on the edge between vertices i and j.

All three measures of entagnlement can be bounded as follows,

$$|M_{max(G)}| \leq E_S(|G\rangle), E_G(|G\rangle), E_R(|G\rangle) \leq |\beta(G)|.$$

Our approach saturates the upper bound. We also assume that we have the canonical form of the graph state $|G\rangle$, that is a state whose underlying graph G has minimal $|M_{max}(G)|$ and $|\beta(G)|$ across the whole LC-equivalency orbit. For a large class of graph states the bounds are equal, hence our method evaluates the entanglement measures exactly. This is true for all bipartite graph states. Important examples of this class of graph states are GHZ states and d-dimensional cluster states. For non-bipartite graph states we have the following conditions for equality of the bounds,

- If $|\alpha(G)| < \frac{N}{2}$ then $|M_{max}(G)| \neq |\beta(G)|$.
- If $|\alpha(G)| > \frac{N}{2}$ then $|M_{max}(G)| = |\beta(G)|$.
- If $|\alpha(G)| = \frac{N}{2}$ then we have the following:
  1. If $M_{max}(G)$ is perfect then $|M_{max}(G)| = |\beta(G)|$.
  2. If $M_{max}(G)$ is not perfect then $|M_{max}(G)| \neq |\beta(G)|$.

Consider the stabilizer $\mathcal{S}$ of a graph state $|G\rangle$. Discard all genertors g_i, where $i \in \beta(G)$. The new Abelian group generated by the remaining generators is labelled $S_\alpha = \langle\{g_i | i \in \alpha(G)\}\rangle$. $\mathcal{S}_\alpha$ stabilizes a $2^{|\beta(G)|}$-dimensional subspace spanned by a set of product states $\{|\psi_i^\alpha\rangle\}$, [2].

The minimal linear decomposition of $|G\rangle$ into product states can be written as $|G\rangle = \frac{1}{\sqrt{D_\alpha}} \sum_{i=1}^{D_\alpha} (-1)^{f_i(\mathcal{S})} |\psi_i^\alpha\rangle$, where $D_\alpha = 2^{|\beta(G)|}$ and $f_i(\mathcal{S})$ is a binary valued function whose value depends on the action of the original stabilizer $\mathcal{S}$ on the states $|\psi_i^\alpha\rangle$. The closest separable state can be written as $\omega = \frac{1}{D_\alpha} \sum_{i=1}^{D_\alpha} |\psi_i^\alpha\rangle\langle\psi_i^\alpha|$. Finally the closest product state can be taken to be any state from the set $\{|\psi_i^\alpha\rangle\}$. Using these states we can readily evaluate the entanglement measures to find out that they give the same value of entanglement,

$$E_S(|G\rangle) = E_G(|G\rangle) = E_R(|G\rangle) = |\beta(G)|.$$

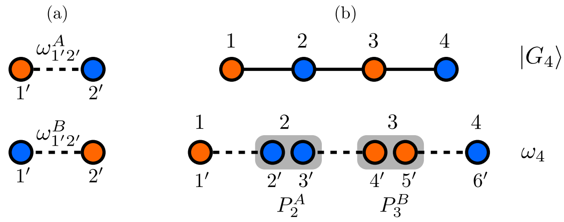

Figure 1: (a) The two 2-qubit states used in the construction of closest separable states. We use the convention that orange vertex corresponds to a qubit in $\{|\pm\rangle\}$ basis and blue vertex corresponds to a qubit in $\{|0\rangle, |1\rangle\}$ basis. (b) Open linear 4-qubit graph state $|G_4\rangle$ and its corresponding closest separable state ω_4.

We stress that the above result holds only in the case when $|M_{max}(G)| = |\beta(G)|$.

Now we focus solely on the CSS ω and we present a novel description of these states based on the PEPS approach. Consider a maximally classically correlated separable state of two virtual qubits ω_2 that is a CSS to 2-qubit graph state. We in fact utilize two such states denoted $\omega^A_{i'j'} := |+0\rangle\langle +0| + |-1\rangle\langle -1|$ and $\omega^B_{i'j'} := |0+\rangle\langle 0+| + |1-\rangle\langle 1-|$. Place a copy of ω_2 on each edge of the graph G, that is replace each physical qubit at vertex a by $|N_a|$ virtual qubits according to the coloring depicted in Fig. (1). Finally we project the virtual qubits at physical site a using $P_a^A := |0\rangle\langle 0\ldots 0| + |1\rangle\langle 1\ldots 1|$ if $a \in \beta(G)$, and $P_a^B := |+\rangle\langle\tilde{+}| + |-\rangle\langle\tilde{-}|$ if $a \in \alpha(G)$. $|\tilde{\pm}\rangle$ is an equal superposition of all tensor product states $\{|+\rangle, |-\rangle\}$ which are ± 1 eigenstates of $X \otimes \ldots \otimes X$.

As the form of general Schmidt decomposition is unknown for weighted graph states with $N \geq 3$ we have to resort to the method of projecting virtual qubits described above to find a candidate CSS. The two-qubit separable state of virtual qubits, $\omega_2(\phi)$, can be easily found. The difficulty lies in identifying suitable projectors that yield a separable state that minimizes the relative entropy.

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