Quantifying non-classical correlations via moment matrix positivity

H. S. Karthik[1] * A. R. Usha Devi[2 3] † A. K. Rajagopal[3 4] ‡ Sudha[5 3] §
J. Prabhu Tej[2] ¶ Andal Narayanan[1] ‖

[1] *Raman Research Institute, Bangalore 560 080, India*
[2] *Department of Physics, Bangalore University, Bangalore-560 056, India*
[3] *Inspire Institute Inc., Alexandria, Virginia, 22303, USA.*
[4] *Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211 019, India.*
[5] *Department of Physics, Kuvempu University, Shankaraghatta, Shimoga-577 451, India.*

Abstract. We investigate if a given set of moments, arising from correlation measurements of three dichotomic observables in the quantum scenario, is compatible with a legitimate grand joint probability distribution. A valid sequence of correlations requires that the corresponding moment matrix is positive. We find an interesting link between moment matrix and the structure of admissible joint probability distribution: *positivity of the moment matrix necessarily enforces that the associated joint probabilities are of the hidden variable form.* Our analysis promotes a quantification of non-classical correlations in terms of the tracenorm of the moment matrix. Examples of spatial and temporal correlations arising within the quantum framework demonstrate our results.

Keywords: joint probabilities, moment matrix, hidden variables, quantum correlations

1 Introduction

Probabilities of measurement outcomes within the quantum framework are fundamentally different from those arising in the classical statistical scenario. This has invoked a wide range of debates on the foundational conflicts about the quantum-classical worldviews of nature [1, 2]. Pioneering investigations by Bell [3], Kochen-Specker [4], Leggett-Garg [5] tied the puzzling quantum features with various no-go theorems. The common aspect underlying the proofs of these no-go theorems points towards the non-existence of a joint probability distribution for the outcomes of all possible measurements performed on a quantum system [2, 3, 4, 5, 6, 7].

On an entirely different perspective, *classical moment problem* [8, 9] addresses the issue of determining a probability distribution given a sequence of statistical moments. Essentially, classical moment problem brings forth that a given sequence of real numbers qualify to be moment sequence of a legitimate probability distribution if and only if the associated moment matrix is positive. In other words, *existence of a valid joint probability distribution* consistent with a given sequence of *moments* gets linked with moment matrix positivity.

In this work, we investigate positivity of 8×8 moment matrix to verify the existence of a valid joint probability distribution of three dichotomic observables in the quantum scenario. This results in an interesting identification: *positivity of the moment matrix implies that the associated joint probabilities assume convex product form.* In other words, hidden variable structure for the joint probabilities emerges naturally – bringing forth a necessary and sufficient condition for non-classicality of correlations via moment matrix positivity criterion. We propose a quantification of non-classical correlations based on the tracenorm of the *normalized* moment matrix. Specific examples of temporal correlations of a single qubit observable at three different times, when the system is evolving (a) under a coherent unitary dynamics and (b) an open system evolution resulting in amplitude damping are examined – with implications on Leggett-Garg macrorealism in terms of moment matrix quantification. We also explore the Bell non-locality, in conjunction with contextuality, in terms of the moment matrix associated with the statistical correlations of three dichotomic observables of a spatially separated two qubit system.

2 Positivity of the moment matrix and the nature of grand joint probabilities

We consider three dichotomic random variables X_1, X_2, X_3. A sequence of eight moments $\{1, \langle X_1\rangle, \langle X_2\rangle, \langle X_3\rangle, \langle X_1X_2\rangle, \langle X_2X_3\rangle, \langle X_1X_3\rangle, \langle X_1X_2X_3\rangle\}$ faithfully encode the details of the joint probability distribution $P(x_1, x_2, x_3)$, $x_i = \pm 1$ [10]. This encryption of trivariate probabilities in these eight moments is reflected in the positivity of the 8×8 moment matrix $M = \langle \xi\, \xi^T\rangle$, where $\xi^T = (1, X_1, X_2, X_3, X_1X_2, X_2X_3, X_1X_3, X_1X_2X_3)$. In other words, given a set of real numbers (which is supposed to be the moment sequence), positivity of the moment matrix ensures that there exists a valid joint probability distribution.

Here, we restrict to the specific case where $\langle X_i\rangle = 0$, $\langle X_1X_2X_3\rangle = 0$. We consider a sequence of eight real numbers $\{1, 0, 0, 0, a, b, c, 0\}$, and raise the question if this set admits a valid classical probability districution $P(x_1, x_2, x_3)$. The corresponding 8×8 moment matrix

*karthik@rri.res.in
†arutth@rediffmail.com
‡attipat.rajagopal@gmail.com
§arss@rediffmail.com
¶prabhutej.j@gmail.com
‖andal@rri.res.in

is given by,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & a & b & c & 0 \\ 0 & 1 & a & c & 0 & 0 & 0 & b \\ 0 & a & 1 & b & 0 & 0 & 0 & c \\ a & c & b & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 1 & c & b & 0 \\ b & 0 & 0 & 0 & c & 1 & a & 0 \\ c & 0 & 0 & 0 & b & a & 1 & 0 \\ 0 & b & c & a & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

The eigenvalues of the moment matrix are readily identified to be $\lambda_{1,2} = 1 + a - b - c, \lambda_{3,4} = 1 - a + b - c, \lambda_{5,6} = 1 - a - b + c$ and $\lambda_{7,8} = 1 + a + b + c$.

By embedding the sequence of numbers in the form of a three qubit density matrix $\rho_{123} = \frac{1}{8}\,[I\otimes I\otimes I + a\,\vec{\sigma}\cdot\hat{n}_1\otimes\vec{\sigma}\cdot\hat{n}_2\otimes I + b\,I\otimes\vec{\sigma}\cdot\hat{n}_2\otimes\vec{\sigma}\cdot\hat{n}_3\ \ + c\,\vec{\sigma}\cdot\hat{n}_1\otimes I\otimes\vec{\sigma}\cdot\hat{n}_3]$ (following the mapping $X_1 \to \vec{\sigma}\cdot\hat{n}_1\otimes I\otimes I,\ X_2 \to I\otimes\vec{\sigma}\cdot\hat{n}_2\otimes I,\ X_3 \to I\otimes\vec{\sigma}\cdot\hat{n}_3\otimes I$), it is easy to see that positivity of the density matrix ρ_{123} matches exactly with that of the moment matrix (1). In other words, we find an interesting connection that the corresponding three qubit density matrix is not a physical one – but is a *pseudo* density matrix when the corresponding sequence does not admit a valid three variable joint probability distribution. (In a recent work [11] it has been highlighted that fitting the temporal correlations of dichotomic quantum observables in the form of a multiqubit density matrix results in a *psuedo* density matrix). However, the nature of joint probability distribution – when the corresponding sequence happens to be a physically valid set – is still left open. Significantly, we find that there is yet another faithful connection between the positivity of the moment matrix and that of a *partially transposed two qubit density matrix*: $\varrho_{12}^{T_1} = \frac{1}{4}\,[I\otimes I - a\,\sigma_x\otimes\sigma_x - b\,\sigma_y\otimes\sigma_y - c\,\sigma_z\otimes\sigma_z]$. Positivity of partial transpose criterion [12, 13] comes to help now – and it ascertains that *admissiblity of a joint probability distribution with the given sequence (moment matrix positivity) is ensured if and only if the associated two qubit density matrix is separable.* This in turn leads to our main identification that the given set of *moments* should necessarily allow a convex product decomposition of the joint probabilities, so as to be declared as a physically valid sequence of moments. A sequence obtained from *all possible* correlation measurement outcomes in the quantum scenario results, in general, in a pseudo moment matrix. A valid sequence of correlations – resulting within the quantum framework – necessarily enforces a hidden variable structure for the grand joint probabilities. A criterion based on the tracenorm of a normalized moment matrix (which agrees identically with the tracenorm of the pseudo three qubit density matrix associated with it) is thus useful to discern non-classsical correlations.

3 Conclusion

We have shown that the question of *existence of a grand joint probabilities* underlying all possible measurements of three dichotomic observables finds an elegant connection with the positivity of the associated moment matrix. Employing well-established results from quantum information theory we prove that positivity of the moment matrix necessarily implies a hidden variable structure for the joint probabilities. We have proposed a criterion based on the tracenorm of the moment matrix to quantify non-classical correlations. Examples of spatial and temporal correlations arising in the quantum scenario illustrate our results.

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