“Pretty strong” converse for the quantum capacity of degradable channels

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Abstract. We exhibit a possible road towards a strong converse for the quantum capacity of degradable channels. In particular, we show that all degradable channels obey what we call a “pretty strong” converse: When the code rate increases above the quantum capacity, the fidelity makes a discontinuous jump from 1 to at most $\frac{1}{\sqrt{2}}$, asymptotically. A similar result can be shown for the private (classical) capacity.

Furthermore, we prove that if the strong converse holds for symmetric channels (which have quantum capacity zero), then degradable channels obey the strong converse: The above-mentioned asymptotic jump of the fidelity at the quantum capacity is then from 1 down to 0.

Keywords: quantum capacity, strong converse, min-entropy

Introduction.—Consider quantum communication via the memoryless quantum channel $\mathcal{N}^{\otimes n}$ (for asymptotically large integer n), given by a completely positive and trace preserving (cptp) map $\mathcal{N} : \mathcal{L}(A') \to \mathcal{L}(B)$, with finite Hilbert spaces A' and B. The quantum capacity $Q(\mathcal{N})$ of $\mathcal{N}$ is defined as the maximum rate at which quantum information can be transmitted asymptotically faithfully over that channel, when using it $n \to \infty$ times.

To make this precise, for a given channel $\mathcal{N}$, we need to have encoding and decoding cptp maps

$$\mathcal{E} : \mathcal{L}(C) \to \mathcal{L}(A'), \quad \mathcal{D} : \mathcal{L}(B) \to \mathcal{L}(C),$$

which together form a *quantum code*; the information to be sent is subjected to the overall effective channel $\mathcal{D} \circ \mathcal{N} \circ \mathcal{E} : \mathcal{L}(C) \to \mathcal{L}(C)$.

There are many ways of defining mathematically the notion that the output is a good approximation of the input, and we refer the reader to the comprehensive treatment of Kretschmann and Werner [2] for a discussion of all the concomitant ways of defining the capacity and the proof that asymptotically and for vanishing error they are the same. Here, we will measure the degree of approximation between states by the fidelity, given as

$$F(\rho, \sigma) := \left\| \sqrt{\rho}\sqrt{\sigma} \right\|_1 = \max |\langle \varphi|\psi\rangle|,$$

where the maximization is over all purifications $|\varphi\rangle$, $|\psi\rangle$ of ρ and σ, respectively. The error is then measured by the purified distance $P(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)^2}$ between any input state ρ and the output state $\sigma = \mathcal{D}(\mathcal{N}(\mathcal{E}(\rho)))$. The maximum dimension $|C|$ of C such that there exists a quantum code for $\mathcal{N}^{\otimes n}$ with error ϵ, is denoted $N(n, \epsilon)$, or more precisely $N(n, \epsilon|\mathcal{N})$ if we want to refer explicitly to the channel. The quantum capacity is now defined as

$$Q(\mathcal{N}) = \inf_{\epsilon>0} \liminf_{n\to\infty} \frac{1}{n} \log N(n, \epsilon).$$

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A Shannon-style formula for the quantum capacity was first given by Lloyd and later proved in full by Shor and then Devetak: more precisely, in these papers the direct (achievability) part was proven; earlier, Schumacher and Nielsen had shown that the same quantity is an upper bound. The formula is given in terms of the *coherent information*

$$I(A\rangle B)_\rho = -S(A|B)_\rho = S(\rho^B) - S(\rho^{AB}),$$

where $S(\rho) = -\mathrm{Tr}\, \rho \log \rho$ is the von Neumann entropy, of a state $\rho^{AB} = (\mathrm{id} \otimes \mathcal{N})\phi^{AA'}$ with a “test state” ϕ on AA'. Namely,

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}),$$

with the single-letter expression

$$Q^{(1)}(\mathcal{N}) = \max_{\phi \in \mathcal{S}(AA')} \big\{ I(A\rangle B)_\rho \text{ s.t. } \rho = (\mathrm{id} \otimes \mathcal{N})\phi \big\}.$$

Weak, strong and “pretty strong” converse.—The fact that the coherent information gives an upper bound on the quantum capacity of general channels has been known since Schumacher and Nielsen. To be precise, they showed that for any entanglement generating code with code space C, for a channel $\mathcal{N} : \mathcal{L}(A') \to \mathcal{L}(B)$ with error ϵ, there exists an input test state $\phi^{AA'}$ such that with $\rho^{AB} = (\mathrm{id} \otimes \mathcal{N})\phi$,

$$(1 - 2\epsilon) \log |C| \le I(A\rangle B)_\rho + 1.$$

Hence, for $\epsilon < \frac{1}{2}$,

$$\frac{1}{n} \log N_E(n, \epsilon) \le \frac{1}{1 - 2\epsilon} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}) + \frac{1}{(1 - 2\epsilon)n}, \quad (1)$$

hence the result that for $n \to \infty$ and $\epsilon \to 0$, the optimal rate cannot exceed $\lim_n \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n})$, which we know is also asymptotically achievable, thanks to Lloyd-Shor-Devetak.

However, for any non-zero $\epsilon > 0$, the upper bound in Eq. (1) is a constant factor away from the capacity, which is the hallmark of a weak converse; it leaves room for a trade-off between communication rate and error, asymptotically.

If the quantum capacity $Q(\mathcal{N})$ is zero, Eq. (1) says something a bit stronger, namely that $N_E(n, \epsilon) \leq O(1)$, at least when $\epsilon < \frac{1}{2}$. We call such a statement *pretty strong converse*, i.e. a proof amounting to

$$\limsup_{n\to\infty} \frac{1}{n} \log N_E(n, \epsilon) \leq Q(\mathcal{N}),$$

at least for error ϵ below some threshold $\epsilon_0 > 0$. A *strong converse* would require the above for all $\epsilon < 1$.

Two simple examples of channels for which the strong converse holds are PPT entanglement binding channels (which have capacity 0) and the ideal channel.

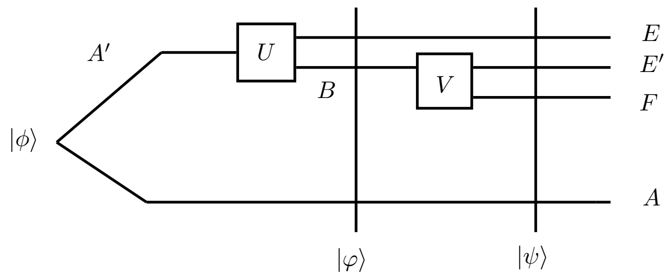

Figure 1: Schematic of a degradable quantum channel, with the input state ϕ between A' and the reference A, the channel output and environment state φ and the state ψ shared between A, F and the two copies of the original environment, E and E'.

Main results.—In all the following statements, $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ is a degradable channel; full proofs are found in [3].

Theorem 1 *For error* $\epsilon < \frac{1}{\sqrt{2}}$,

$$\log N(n, \epsilon) \leq \log N_E(n, \epsilon) \leq nQ^{(1)}(\mathcal{N}) + O\left(\sqrt{n \log n}\right).$$

Together with the direct part we thus get:

Corollary 2

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \log N(n, \epsilon) = \lim_{n\to\infty} \frac{1}{n} \log N_E(n, \epsilon),$$

for any $0 < \epsilon < \frac{1}{\sqrt{2}}$. *Compared to the original definition this is simpler as we do not need to vary* ϵ*, and there is convergence rather than reference to* $\liminf$ *or* $\limsup$.

Note that the error $\frac{1}{\sqrt{2}}$ is precisely that achieved asymptotically by a single 50%-50% erasure channel acting on the code space, and of other suitable symmetric (i.e., degradable and anti-degradable) channels.

For a given channel $\mathcal{N}$, we denote the largest number M of messages such that there exists a private classical code with error ϵ and privacy δ (which is itself defined in terms of the fidelity of the complementary channel $\mathcal{N}^c$, see [3]) by $M(n, \epsilon, \delta)$. The *(weak) private capacity* of $\mathcal{N}$ is then defined as

$$P(\mathcal{N}) = \inf_{\epsilon,\delta>0} \liminf_{n\to\infty} \frac{1}{n} \log M(n, \epsilon, \delta).$$

A capacity formula for P was determined by Devetak and Cai, Winter and Yeung; like Q it is only known as a regularized characterization in general. By the monogamy of entanglement, we know that $P(\mathcal{N}) \geq Q(\mathcal{N})$, but in general this inequality is strict. However for degradable channels, it was proved by Smith that the private capacity $P(\mathcal{N})$ equals the quantum capacity $Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$, and is hence given by a simple single-letter formula.

Theorem 3 *For error* ϵ *and privacy* δ *such that* $\epsilon + 2\delta < \frac{1}{\sqrt{2}}$ *(e.g.* $\epsilon = \delta < \frac{1}{3\sqrt{2}} \approx .2357$*),*

$$\log M(n, \epsilon, \delta) \leq nQ^{(1)}(\mathcal{N}) + O\left(\sqrt{n \log n}\right).$$

Corollary 4 *For any* $\epsilon, \delta > 0$ *such that* $\epsilon + 2\delta < \frac{1}{\sqrt{2}}$,

$$P(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \log M(n, \epsilon, \delta).$$

Theorem 5 *Let* $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ *be a degradable channel, denoting its environment by* E*. Then there is an associated symmetric channel* $\mathcal{M}$*, such that* $\mathcal{N}$ *obeys the strong converse for its quantum capacity, if* $\mathcal{M}$ *does (note that by the no-cloning argument,* $Q(\mathcal{M}) = 0$*). More precisely, there exists a constant* μ *such that with* $\lambda = \frac{1-\epsilon}{5}$,

$$\log N_E(n, \epsilon|\mathcal{N}) \leq nQ^{(1)}(\mathcal{N}) + \mu\sqrt{n \ln \frac{64n^{|A|^2}}{\lambda^2}} + 8 \log \frac{1}{\lambda} + \log N_E(n, 1-\lambda|\mathcal{M}).$$

Conclusion.—While we have shown results for degradable channels, we note that most channels are of course not degradable (nor anti-degradable). For practically all these others we do not have any approach to obtain a strong or even a pretty strong converse. One might speculate that other channels with additive coherent information, hence with a single-letter capacity formula, are amenable, to our method. But already the very attractive-looking class of *conjugate degradable* [1] channels poses new difficulties.

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