Generalized monogamy of contextual inequalities from the no-disturbance principle

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Abstract. We prove a monogamy relation for contextual inequalities by exploiting the principle of no-disturbance. This implies that monogamy of quantum correlation exists beyond multipartite scenarios. A necessary and sufficient condition is given using graph theoretic techniques to identify a set of mutually exclusive measurements that exhibit monogamy.

Keywords: monogamy, contextual inequalities, graph theory

1 Motivation and summary

It is known that we cannot assign a value to all observables in quantum mechanics, which has an observable consequence when there are several measurements that can be jointly performed [1]. For example, suppose we have four measurements A, B, C, and D, out of which the pairs (A, B), (B, C), (C, D), and (D, A) can be jointly measured. The outcomes of a jointly performable measurements yield a joint probability distribution such as $p(a, b)$ for A and B, where the lower case letters shall indicate the outcomes of the respective measurements. Let each measurement return value 1 or -1. If the outcomes of all the measurements were determined prior to the time of measurement, the probability distributions $p(a, b)$, $p(b, c)$, $p(c, d)$ and $p(d, a)$ must satisfy the Clauser-Horne-Shimony-Holt (CHSH) inequality [2], $\left|\sum_{a,b,c,d=1,-1} ab \cdot p(a, b) + bc \cdot p(b, c) + cd \cdot p(c, d) - da \cdot p(d, a)\right| \leq 2$. In quantum mechanics, the CHSH inequality can be violated up to $2\sqrt{2}$, which is a signature of quantum correlation. The CHSH inequality applies to any physical setup where there are at least four dichotomic observables satisfying the said joint measurablility. Hence, for some physical system with more than four measurements, multiple CHSH inequalities can be defined, but this does not imply that all can be violated simultaneously. In fact, such simultaneous violation for certain combinations of CHSH inequalities is prohibited.

The impossibility of the simultaneous violation implies that quantum correlation exhibits a monogamous behavior. Monogamy of quantum correlation for other Bell inequalities is also known [3] and found to be useful in secure quantum key distribution [4] and interactive proof systems [5]. Applications of monogamies go beyond these multi-party protocols [6, 7].

Despite its usefulness, the monogamy based on Bell inequalities applies only to spacelike separated measurements. For other types of measurements, quantum correlations can be identified by a broader class of inequality known as *contextual inequalities* [8]. In this work, we prove that monogamy also exists for quantum correlations of general measurements by exploiting contextual inequalities. Our proof of monogamy is based on the principle of no-disturbance, which asserts that jointly performable measurements should not influence each other. We also formulate a necessary and sufficient condition for a set of measurements to admit monogamy using graph theoretic techniques.

2 Monogamy of KCBS-type inequalities

The Klyachko-Can-Binicoglu-Shumovsky (KCBS) inequality [8] is the simplest contextual inequality that was introduced to test a nonclassical feature of a single (three-level) system. The KCBS inequality uses five measurements $A_1, \ldots, A_5$ with outcomes $a_i = 0, 1$. These measurements are cyclically compatible (i.e., joint probability distributions $p(a_i, a_{i+1})$ exist for $i = 1, \ldots, 5$, where a_6 is identified with a_1) and exclusive (i.e., $a_i a_{i+1} = 0$). These measurements can be represented by the "commutation graph" corresponding to a pentagon, where the vertices of the pentagon graph represent the five measurements and edges between any two vertices indicate that the two corresponding measurements can be jointly performed and are mutually exclusive (see the pentagons in Fig. 1). The KCBS inequality reads, $\sum_{i=1}^{5} p(A_i = 1) \leq 2$.

Consider two sets of cyclically compatible and exclusive measurements $\{A_i\}$ and $\{A'_i\}$. Each set gives rise to a KCBS inequality. Let us assume that the triples (A_1, A'_1, A'_2) and (A_4, A_5, A'_5) are each jointly measurable and mutually exclusive. This scenario is represented by the commutation graph in Fig. 1. Therefore, in addition to $p(a_i, a_{i+1})$ and $p(a'_i, a'_{i+1})$, one can experimentally determine probabilities $p(a_1, a'_1, a'_2)$ and $p(a'_5, a_4, a_5)$.

We then introduce the principle of no-disturbance, which is formulated as follows. Suppose that one can perform several different measurements A, B, C, etc. Assume that measurement pairs (A, B) and (A, C) can be jointly performed, which implies the existence of the joint probabilities $p(a, b)$ and $p(a, c)$. The principle of no-disturbance is then the condition that the marginal

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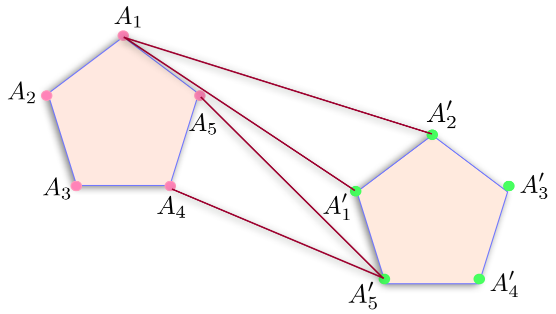

Figure 1: Graphical representation of two KCBS inequalities that satisfies the monogamy relation.

probability $p(a)$ calculated from $p(a,b)$ is the same as that of calculated from $p(a,c)$, i.e., $\sum_b p(A=a,B=b)=\sum_c p(A=a,C=c)=p(A=a)$. In the KCBS scenario, the no-disturbance principle is satisfied by setting $p(A_1=1)=p$ and $p(A'_5=1)=q$ for all measurement combinations that are jointly measurable and include A_1 or A'_5. Mutual exclusiveness implies that $p(A'_1=1)+p(A'_2=1)\leq 1-p$ and $p(A_4=1)+p(A_5=1)\leq 1-q$ in addition to $p(A_i=1)+p(A_{i+1}=1)\leq 1$ and $p(A'_i=1)+p(A'_{i+1}=1)\leq 1$, together which imply the monogamy relation

$$\sum_{i=1}^{5} p(A_i=1)+\sum_{i=1}^{5} p(A'_i=1)\leq 4. \tag{1}$$

Clearly, only one KCBS inequality out of the two sets $\{A_i\}$ and $\{A'_i\}$ can be violated. Similar monogamies hold for any inequalities of this kind [9].

Having illustrated the method for deriving monogamy relations for contextual inequalities we now proceed to formulate it using some graph-theoretic notions.

Proposition 1 *A commutation graph G representing a set of n measurements (for any n) admits a joint probability distribution for these measurements if it is a chordal graph.*

A chordal graph is a graph that does not contain an induced cycle of length greater than 3.

We now proceed to explicitly identify the commutation graphs that give rise to monogamy relations for a given set of n KCBS-type contextual inequalities (with classical bound R).

Proposition 2 *Consider a commutation graph representing a set of n KCBS-type contextual inequalities each of which has classical bound R. Then this graph gives rise to a monogamy relation using the outlined method if and only if its vertex clique cover number is $n*R$.*

The vertex clique cover number is the minimal number of cliques required to cover all the vertices of the graph. The above Proposition can be extended to the case when when one is interested in the monogamy of a set of n_k different contextual inequalities with different classical bounds R_k, with $\sum_k n_k = n$. Then the condition becomes that the vertex clique cover number equal $\sum_k n_k R_k$.

The argument so far proves that the monogamy relation (1) exists provided that there are measurements obeying the constraints represented by the commutation graph. In Ref. [10], we present a set of projectors in a four-dimensional real space that meets these constraints.

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