Entanglement between two subsystems, the Wigner semicircle and extreme value statistics.

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Abstract. The entanglement between two arbitrary subsystems of random pure states is studied via properties of the density matrix's partial transpose, $\rho_{12}^{T_2}$. The density of states of $\rho_{12}^{T_2}$ is found to be asymptotically the Wigner semicircle. A simple random matrix model is found to capture these properties well, including a NPT–PPT transition. The smallest eigenvalue of $\rho_{12}^{T_2}$ is found to follow the extreme value statistics of random matrices. Thus the Tracy-Widom distribution is used to calculate the fraction of entangled states at critical dimensions. These results are then tested in a quantum dynamical system of three coupled standard maps.

Keywords: Quantum Entanglement, Quantum Information and Random Matrix Theory.

A random state corresponds to assuming minimal prior information about the quantum system. Thus they are very general and can be seen as typical states. Eigenstates of a nonintegrable Hamiltonian can often result in states whose statistical properties closely resemble those of random states. Previous results suggest that almost all pure states have almost maximal bipartite entanglement [1, 2] and are shown to be useful for quantum information processing [3]. Very little is known in the case of entanglement between two subsystems of tripartite (three parts) random pure states. An example is provided by two interacting systems which are kept in a heat bath and together they are in a pure state.

Consider random tripartite pure states in $\mathcal{H}_{2^{L_1}} \otimes \mathcal{H}_{2^{L_2}} \otimes \mathcal{H}_{2^{L-L_1-L_2}}$, having L qubits and a partition into three subsystems of L_1, L_2 and $L - L_1 - L_2$ qubits. We have studied [4] entanglement between the two subsystems whose density matrix is denoted by ρ_{12}, using the properties of its partial transpose (PT), $\rho_{12}^{T_2}$. It is found that the eigenvalue density of $\rho_{12}^{T_2}$ is close to the Wigner semicircle law for sufficiently large L_1, L_2 and $L_1 + L_2 \ll L$. Wigner semicircle law is an important distribution of random matrix theory, for example it is the eigenvalue density of the various Gaussian ensembles [5]. It is found that typically the two subsystems are entangled if $L_1 + L_2 > L/2 - 1$ and typically PPT if $L_1 + L_2 < L/2 - 1$. The case when $L_1 + L_2 = L/2 - 1$ is critical, with a coexistence of NPT and PPT states even for large L.

A simple random matrix model for the partial transpose of the density matrix of subsystems in a typical pure states is studied. This is found to capture these entanglement properties, including that of the critical case. Log negativity is used as a measure to quantify the typical entanglement between the subsystems and analytic formulas for this are derived based on the simple model. It is found that the eigenvalue density of $\rho_{12}^{T_2}$ is a skewed distribution. Thus the skewness of the eigenvalue density of $\rho_{12}^{T_2}$ is derived analytically, using the average of the third moment over the ensemble of random pure states. The third moment after partial transpose is also shown to be related to a generalization of the Kempe invariant and is invariant under permutation of subsystems for any given state.

It is found that the smallest eigenvalue after partial transpose follow the extreme value statistics of random matrices, namely the Tracy-Widom distribution [6, 7]. This distribution, with relevant parameters obtained from the model, is found to be useful in calculating the fraction of entangled states at critical dimensions. These results are tested in a quantum dynamical system of three coupled standard maps, where one finds that if the parameters represent a strongly chaotic system, the results are close to those of random states, although there are some systematic deviations at critical dimensions.

References

[1] P. Hayden, D. W. Leung, and A. Winter, Commun. Math. Phys. 265, 95 (2006).

[2] J. N. Bandyopadhyay and A. Lakshminarayan, Phys. Rev. Lett. 89, 060402 (2002).

[3] L. Masanes, Phys. Rev. Lett. 96, 150501 (2006).

[4] U. T. Bhosale, S. Tomsovic, and A. Lakshminarayan, Phys. Rev. A 85, 062331 (2012).

[5] M. L. Mehta, *Random Matrices* (Elsevier Academic Press, 3rd Edition, London, 2004).

[6] C. Tracy and H. Widom, Commun. Math. Phys. 159, 151 (1994).

[7] C. Tracy and H. Widom, Commun. Math. Phys. 177, 727 (1996).

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