Construction and Physical Realization of POVMs on a symmetric subspace

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Abstract. Positive Operator Valued Measures (POVMs) are of great importance in quantum information and quantum computation. Using the well-known spherical tensor operators, we propose a scheme of constructing a set of $N-$qubit POVMs operating on the symmetric state space of dimension $N+1$ which is a subspace of the 2^N dimensional Hilbert space. By invoking Neumark's theorem we show for a 2-qubit case, that POVM's in the 3 dimensional symmetric space can be physically realized as projection operators in the larger 4-dimensional Hilbert space.

Keywords: POVM, quantum information, quantum computation, spherical tensors

1 Introduction

Positive-Operator-Valued-Measures(POVMs), are the most general class of quantum measurements, of which von Neumann measurements are merely a special case. The rapidly developing quantum information theory has generated a lot of interest in the construction and experimental implementation of POVMs [1]. Some tasks that cannot be performed using projective measurements can be completed using POVMs. For example, a set of non-orthogonal states cannot be distinguished using projective measurements, but can be discriminated unambiguously using POVMs [2]. This concept can be used to construct POVMs in conclusive teleportation [3]. Recently, POVMs instead of usual projective measurements has been utilized to realize some quantum communication protocols [4]. In the context of Remote State Preparation (RSP), POVM has already attracted a lot of attention [5]. Recently it has been shown [6] how quantum filtering can be performed in the POVM setting.

In this paper we introduce the most general method of constructing POVMs in terms of detection operators, on $N+1$ dimensional symmetric space which is a subspace of 2^N dimensional Hilbert space in which N qubits reside. A set of N-qubit pure states that respect permutational symmetry are called symmetric states. Symmetric space can be considered to be spanned by the eigen states $|jm>, -j \leq m \leq +j$ of angular momentum operators J^2 and J_z, where $j = N/2$. A large number of experimentally relevant states [7] possesses symmetry under particle exchange, which significantly reduces the computational complexity. The detection operators are found to be the well known irreducible tensor operators τ^k_q which are used for the standard representation [8] of spin density matrices. Further, by invoking Neumark's theorem [9], we demonstrate the physical implementation of our symmetric two-qubit POVMs as projection operators in the two qubit tensor product space.

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2 Operator representation in terms of irreducible tensors

The systematic use of tensor operators was first suggested by Fano[8]. In this representation, any Hermitian matrix 'H' in the symmetric space of dimension $N+1$ in terms of spherical tensor parameters can be expressed as

$$H(\vec{J}) = \sum_{k=0}^{2j} \sum_{q=-k}^{+k} h^k_q \, \tau^{k^\dagger}_q(\vec{J}) \tag{1}$$

where $\tau^k_q\,'s$ (with $\tau^0_0 = I$, the identity operator) are the irreducible (spherical) tensor operators of rank 'k' in the $(2j+1)$ dimensional spin space. The operators $\tau^k_q\,'s$ of rank 'k' and projection 'q' are homogeneous polynomials constructed out of angular momentum operators J_x, J_y, J_z. In particular, following the well known Weyl construction [10] for $\tau^k_q\,'s$ in terms of angular momentum operators, we have

$$\tau^k_q(\vec{J}) = \mathcal{N}_{kj} \, (\vec{J} \cdot \vec{\bigtriangledown})^k \, r^k \, Y^k_q(\hat{r}) \,, \tag{2}$$

where $\mathcal{N}_{kj}$ are the normalization factors and $Y^k_q(\hat{r})$ are the spherical harmonics.

The $\tau^k_q\,'s$ satisfy the orthogonality relations

$$Tr(\tau^{k^\dagger}_q \tau^{k'}_{q'}) = (2j+1) \, \delta_{kk'} \delta_{qq'} \tag{3}$$

and

$$\tau^{k^\dagger}_q = (-1)^q \tau^k_{-q} \,,$$

Here the normalization has been chosen so as to be in agreement with Madison convention [11]. The matrix elements of the tensor operator are given by

$$\langle jm'|\tau^k_q(\vec{J})|jm\rangle = \sqrt{2k+1} \; C(jkj; mqm') \tag{4}$$

where $C(jkj; mqm')$ are the Clebsch-Gordan coefficients.

3 Positive Operator valued Measure

The spherical tensor operators can be used as detection operators in constructing a set of POVMs for a symmetric space as

$$E_q^k = \frac{\tau_q^k \tau_q^{k^\dagger}}{N^2}, \tag{5}$$

satisfying the properties of partition of unity, hermiticity and positivity.

3.1 Partition of unity

Since E_q^k's add up to unity i.e., $\sum_{kq} E_q^k = 1$, we have

$$\langle jm| \sum_{kq} \frac{\tau_q^k \tau_q^{k^\dagger}}{N^2} |jm'\rangle = \delta_{mm'} \tag{6}$$

with

$$N^2 = \sum_{k'q'} [k']^2 C(jk'j; m-q'\, q'm)^2 \tag{7}$$

3.2 Hermiticity

Since

$$(\tau_q^k \tau_q^{k^\dagger})^\dagger = (\tau_q^{k^\dagger})^\dagger \tau_q^{k^\dagger} = \tau_q^k \tau_q^{k^\dagger}, \tag{8}$$

$E_q^{k^\dagger} = E_q^k$ for all k, q

3.3 Positivity

We can also show that $\langle\psi|E_q^k|\psi\rangle \geq 0$ for all k, q and all $\langle\psi| \in H$, i.e., E_q^k are positive operators.

It can be shown that the above POVMs do not form an orthogonal set.
If we perform the measurement and do not record the results then the post measurement state is described by the density operator,

$$\rho^f = E_q^k \rho^i E_q^k \tag{9}$$

4 Neumark's Theorem

Next, we address the question, can a POVM E_q^k acting on a symmetric space, be interpreted as resulting from a measurement on a larger space? The answer is yes. For eg., Consider one of the POVM's, namely

$$E_1^1 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

in the symmetric $|1m\rangle$ basis, $m = 1, 0, -1$. The relationship between $|1m >$ basis and the computational basis is such that $|11\rangle = |\uparrow\uparrow\rangle$, $|10\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$ and $|1-1\rangle = |\downarrow\downarrow\rangle$. The Matrix representation of E_1^1 in the 2-qubit state space of dimension 4, in the computational basis $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$ is given by

$$\epsilon_1^1 = U^\dagger (E_1^1 \oplus 0) U, \tag{11}$$

where $\oplus$ represents the direct sum and

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{12}$$

is the unitary matrix which transforms computational basis to the angular momentum basis $|11\rangle, |10\rangle, |1-1\rangle, |00\rangle$. Thus

$$\epsilon_1^1|\psi\rangle = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \frac{1}{3}\begin{pmatrix} a \\ \frac{b+c}{2} \\ \frac{b+c}{2} \\ 0 \end{pmatrix} \tag{13}$$

where

$$|\psi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle; \quad |a|^2+|b|^2+|c|^2+|d|^2 = 1 \tag{14}$$

is the most general pure state in the 2-qubit state space. Observe that the resultant state is a symmetric state and is given by $\frac{1}{3}a|11\rangle + \frac{1}{3\sqrt{2}}(b+c)|10\rangle$. Thus E_q^k's project vectors in the 4-dimensional Hilbert space onto 3-dimensional symmetric space. Note that ϵ_1^1 can be expressed in terms of the well-known Pauli spin matrices, given by

$$\begin{aligned}\epsilon_1^1 = \frac{1}{6}\big[&(I_1 \otimes I_2) + \frac{1}{2}(\sigma_z(1) \otimes I_2) + \frac{1}{2}(I_1 \otimes \sigma_z(2)) \\ &+ \frac{1}{2}(\sigma_x(1) \otimes \sigma_x(2)) + \frac{1}{2}(\sigma_y(1) \otimes \sigma_y(2))\big].\end{aligned} \tag{15}$$

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