On the Volume of Separable Bipartite States

Rajeev Singh[1] * Ravi Kunjwal[1] R. Simon[1]

[1] *Optics & Quantum Information Group,*
The Institute of Mathematical Sciences,
CIT Campus, Taramani, Chennai 600113, India.

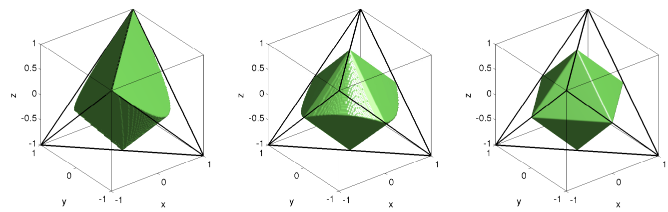
Abstract. Every choice of orthonormal frame in the Hilbert space of a bipartite system corresponds to one set of all mutually commuting density matrices or, equivalently, to the classical state space of the system; the quantum state space itself can thus be profitably viewed as an orbit of classical state spaces, one copy for each orthonormal frame. We exploit this connection to study the relative volume of separable states of a quantum bipartite system. While the two-qubit case is analysed in considerable detail, for higher dimensional systems we fall back on Monte Carlo.

Keywords: Separable Bipartite States, Geometry of State Space, Generalized Bloch sphere

States of a quantum system are represented by density operators (positive semidefinite unit-trace operators acting on a Hilbert space of dimension d). The set of all density operators of a system constitutes a convex subset of $\mathbb{R}^{d^2-1}$; this is the *state space* of the system. An understanding of the geometry of the state space is of fundamental importance. The state space of a two-level system or qubit is the well-known Bloch (or Poincaré) sphere, while the generalized Bloch sphere of higher dimensional system is much richer, and more complex to visualize and analyze. When d is non-prime, it is possible that the system is composite, i.e. made up of two or more subsystems. For example a 4 dimensional system could be a single quantum system with four levels or classical states, or a pair of two-level systems (qubits). In the case of composite systems, the issue of separability becomes important, entanglement being a characteristic feature of quantum theory, and a key resource in quantum information processing.

During a recent reading of the seminal paper by K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein [1] on this subject—a paper which we had read several times in the past—the following passage therein captured our attention: *"Our numerical results agree with these bounds, but to our surprise the probability that a mixed state $\rho \in \mathcal{H}_2 \times \mathcal{H}_2$ is separable exceeds 50%."* Their paper established an interesting analytical lower bound for the probability of separability (fractional volume of separable states) of a two-qubit system to be 0.302, and on numerical (Monte Carlo) estimation they found it to be definitely above 50%. An attempt to remove this 'surprise' element was the humble beginning of the study whose results are reported here.

Compared to the quantum state space, the classical (statistical) state space of a d-state system is extremely simple. Indeed, it is the regular simplex $\Delta_{d-1} \subset \mathbb{R}^{d-1}$, the convex body defined by d equidistant vertices. *The quantum state space itself can be viewed as the union of an orbit of simplices Δ_{d-1}*, and this fact is fundamental to both our point of view and analysis. Mutually commuting density operators constitute a simplex,



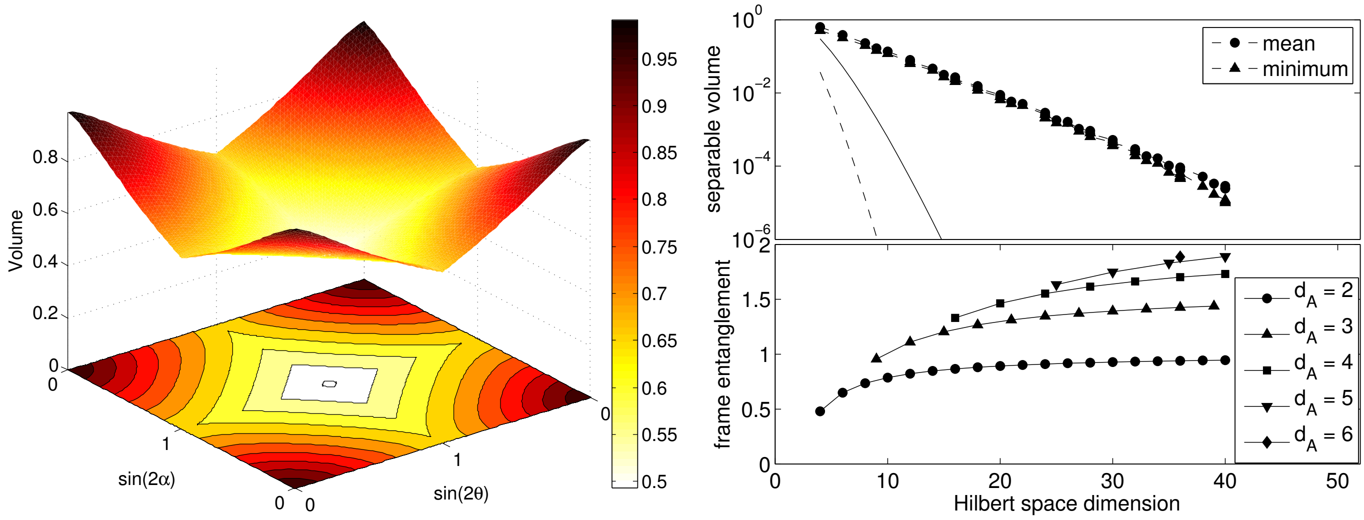
(a) $(0, \pi/8)$ (b) $(\pi/8, \pi/8)$ (c) $(\pi/4, \pi/4)$

Figure 1: Separable regions for different values of (θ, α). The tetrahedron represents the set of all density matrices with same eigenvectors. The volume enclosed by the shaded surface shows the separable region for the given frame. We find that the separable set is the entire tetrahedron for $(\theta, \alpha) = (0, 0)$ and is an octahedron for $(\theta, \alpha) = (\pi/4, \pi/4)$ as expected. For other values of (θ, α) we find the separable set to be the tetrahedron limited by planes and conic surfaces.

and change of basis [determined by an unitary matrix $U \in SU(d)$] labels the points along the orbit of simplices. It is true that simplices at two distinct points on the orbit are not necessarily non-overlapping. But such overlap is obviously of zero measure, since only density operators with degenerate spectra can sit in two distinct simplices.

For a $d_1 \times d_2$ system the relevant simplex is $\Delta_{d_1 d_2 - 1}$, the dimension of the orbit is $d_1 d_2 (d_1 d_2 - 1)$. *Separability issues are invariant under local unitaries $U_{d_1} \times U_{d_2}$*, so it is sufficient to restrict attention to locally inequivalent simplices. This removes $d_1^2 + d_2^2 - 2$ parameters, and the orbit of locally inequivalent simplices has dimension $(d_1^2 - 1)(d_2^2 - 1) - d_1 d_2 + 1$. We begin with the two-qubit system, so the simplex Δ_3 is the tetrahedron. Each choice of orthonormal basis, *frame* hereafter, corresponds to a Δ_3 and we are interested in frames which are not locally equivalent—a 6-parameter family. The fractional separable volume $f = V_{sep}/V_{tot}$ is computed for each frame or tetrahedron and is shown to be in the range $0.5 \leq f \leq 1$, the lower limit 0.5 obtaining if and only if all the four vectors of the frame are maximally entangled, and for the upper limit they are products. It thus becomes 'obvious' that the ensemble average of fractional volume would be

*rajeev@imsc.res.in



(a) (b)

Figure 2: (a) Volume of separable states as a function of the two entanglement parameters $\sin 2\theta, \sin 2\alpha$ in the case of the two-parameter family. We find that the volume is minimum for $(\pi/4, \pi/4)$ which corresponds to the Bell-diagonal frame. (b) Top panel shows mean and minimum of separable volume over frames as a function of Hilbert space dimension, showing an exponential decrease. It also shows the lower bounds given by [1] as solid line, and that given by [2] as dashed line. Bottom panel shows the corresponding mean frame entanglement with different symbols for different d_A for fixed $d_A d_B$.

strictly greater than 0.5.

For clarity of presentation, we begin with a special two-parameter family of locally inequivalent frames which are obtained as linear combinations of the computational basis pair $\{|00\rangle, |11\rangle\}$ and the pair $\{|01\rangle, |10\rangle\}$, *with no superposition across pairs*:

$$\begin{aligned}
|\Psi_1\rangle &= \cos\theta\ |00\rangle + \sin\theta\ |11\rangle, \\
|\Psi_2\rangle &= \sin\theta\ |00\rangle - \cos\theta\ |11\rangle, \\
|\Psi_3\rangle &= \cos\alpha\ |01\rangle + \sin\alpha\ |10\rangle, \\
|\Psi_4\rangle &= \sin\alpha\ |01\rangle - \cos\alpha\ |10\rangle.
\end{aligned} \tag{1}$$

These frames can be viewed, in an obvious manner, as a two-parameter generalization of the Bell or magic frame of maximally entangled states. In Fig. 1 we picture the separable region (inside the tetrahedron) for a few values of (θ, α). The boundaries of the separable region consist entirely of conic surfaces and planes, and this fact has simple analytic basis. The Bell or magic frame corresponds to $\theta = \alpha = \pi/4$ (as is well known the separable region is an octahedron in this case), also shown in the figure. We numerically calculate the volume of the separable region for all values of (θ, α), and the result is pictured in Fig. 2 (a). The volume decreases with increasing 'entanglement of the frame'.

A canonical parameterization of the full six-parameter family of locally inequivalent frames, as also the distribution of fractional volume of separable states over these frames, has been obtained.

To gain quick insight into the situation in respect of higher dimensional systems we perform Monte Carlo sampling following the scheme in [1]. We show in Fig. 2 (b) the mean and minimum separable volume and mean frame entanglement as a function of Hilbert space dimension. Consistent with earlier work [1], we find that the separable volume decreases exponentially with Hilbert space dimension. However we point out that this exponential decrease in the volume of separable states with increasing Hilbert space dimension implies an *increase* in 'effective radius' for separable states. This provides some new insights, as earlier results have claimed a decreasing lower bound on this effective radius [3]. More importantly, there exists one claim that *an upper bound* on this effective radius also decreases with increasing Hilbert space dimension [3] for the case of quantum systems composed of many qubits.

Our approach generalizes to higher dimensional systems, wherein qualitatively new features emerge. For instance, for the qutrit-qutrit systems not all frames of maximally entangled states are locally equivalent and, consequently, they lead to unequal fractional volume of separable states and, perhaps surprisingly, the Bell frame is not the one to result in minimum separable volume.

References

[1] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, "Volume of the set of separable states," *Phys. Rev. A*, vol. 58, no. 2, pp. 883–892, 1998.

[2] G. Vidal and R. Tarrach, "Robustness of entanglement," *Phys. Rev. A*, vol. 59, no. 1, pp. 141–155, 1999.

[3] S. Szarek, "Volume of separable states is super-doubly-exponentially small in the number of qubits," *Phys. Rev. A*, vol. 72, no. 3, p. 032304, 2005.