Positive maps and quantum filtration

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Abstract. The discovery of Bell inequality showed that quantum correlations cannot be modeled using local hidden variables. Quantum entanglement is necessary for the violation of Bell inequalities. However, given the fact that entanglement can come in may forms and one may need positive maps which are not completely positive for its characterization the exact relationship between Bells inequalities and entangled states is not fully discovered. Gisin showed that for some of such states violation can be created by local filters. In this work we show a connection between increasing the violation of Bell inequalities and local filters positive map which can detect the entanglement of that states.

Keywords: Entanglement, Bell inequality

1 Introduction

Existence of quantum entanglement is one of the fundamental theme in quantum world. This is exploited heavily in quantum computation [9]. A physical system is represented by a complex Hilbert space denoted by $\mathcal{H}$. We consider only finite dimensional situations, i.e. $\mathcal{H} = \mathbb{C}^n$. The physical states of the system are represented by $\rho \in \mathcal{B}(\mathcal{H})$ which is a positive definite self-adjoint operator with unit trace (a trace class operator). A rank 1 state is called pure state, otherwise it is mixed. The set of states forms a convex set and the extremal points of this set are pure states which are rank 1 operators.

For composite systems, the Hilbert space is the tensor product of the Hilbert spaces of the individual systems. State space of a bipartite system is given by $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. A bipartite state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is called a separable state if it can be written as

$$\rho = \sum_{j=1}^{n} p_j \rho_j^A \otimes \rho_j^B, \quad p_j > 0, \quad \sum_{j=1}^{n} p_j = 1. \tag{1}$$

where ρ_j^A and ρ_j^B are states in the systems A and B respectively. If a state $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ cannot be written in the above form, then it is an entangled state.[1]

The central problem in quantum information theory is that given an arbitrary (bipartite) state, determine whether it is entangled or separable. In general the problem is NP-hard [6]. To attack this, there are different approaches based on Bell's inequality, positive maps, covariance matrices and so on. For a survey, see [5, 7].

In this poster, we use the concept of positive maps which are not completely positive. We combine it with an invertible super-positive map and construct new classes of witnesses. In this process, we show that, applying such a super-positive map is similar to the concept of *quantum filtration* considered by Gisin [4].

We give the necessary definitions in the section 2 and in the section 3 we give the results.

2 Positive maps: in a nutshell

Let us consider the two qubit state

$$\rho(\lambda, \alpha) = \lambda P_{\psi_{\alpha,\beta}} + \frac{1}{2}(1-\lambda)(P_{\psi_{++}} + P_{\psi_{--}}); \tag{2}$$

where

$$|\psi_{\alpha,\beta}\rangle = \alpha|+-\rangle - \beta|-+\rangle, \quad \psi_{++} = |++\rangle, \quad \psi_{--} = |--\rangle; \tag{3}$$

and $\alpha > \beta > 0$ are real numbers with $\alpha^2 + \beta^2 = 1$; where $|+\rangle$ and $|-\rangle$ have their usual meanings. It can be shown that when

$$\lambda \le \frac{1}{1+\alpha^2\beta^2}, \tag{4}$$

Bell inequality [1, 2] is not violated. Using concepts of polarising optical fibers, Gisin [4] used the transformation matrix $T = \begin{pmatrix} \sqrt{\frac{\beta}{\alpha}} & 0 \\ 0 & 1 \end{pmatrix}$, we get a state $\rho_{\text{filtered}} = T\rho(\lambda, \alpha)T^\dagger$, which violates Bell-CHSH inequality if $\lambda > \frac{1}{1+2\alpha\beta(\sqrt{2}-1)}$. Hence by using an invertible map a local state is converted to a nonlocal state. Borrowing terminologies from quantum optics, such an operation is called quantum filter.

It is interesting to note that a similar concept appearers in the theory of positive maps and detection of entanglement. We give a few definitions in this field.

Let $\mathcal{H}$, $\mathcal{K}$ be (finite dimensional) Hilbert spaces. For $\mathcal{B}(\mathcal{H})$, let us denote $\mathcal{B}(\mathcal{H})_h$ to be the set of all Hermitian operators and $\mathcal{B}(\mathcal{H})_+$ be the set of all positive semidefinite objects. A map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is said to be a positive map (P), if $\varphi(\mathcal{B}(\mathcal{H})_+) \subseteq \mathcal{B}(\mathcal{K})_+$. φ is said to be k-positive (k-P) if the natural extension

$$\begin{aligned} \mathbb{1}_k \otimes \varphi : \mathcal{M}_k(\mathcal{B}(\mathcal{H})) &\to \mathcal{M}_k(\mathcal{B}(\mathcal{K})) \\ ((a_{i,j})) &\mapsto ((\varphi(a_{i,j}))), \quad a_{i,j} \in \mathcal{B}(\mathcal{H}); \end{aligned}$$

is a positive map. It is called completely positive (CP) if it is k-P for all $k \ge 1$. $(\mathcal{B}(\mathbb{C}^k \otimes \mathcal{H}) \simeq \mathcal{B}(\mathbb{C}^k) \otimes \mathcal{B}(\mathcal{H}) \simeq$

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[1]The above definitions can easily be extended to multipartite systems. However, in this work, we confine ourselves in bipartite cases only.

$\mathcal{M}_k(\mathcal{B}(\mathcal{H}))$ denotes the set of all $k \times k$ matrices with entries in $\mathcal{B}(\mathcal{H})$). The structure of CP maps are given by Kraus representation [3, 8, 11]. A CP map is called d super-positive, if its minimal Kraus representation consists of Kraus operators of (matrix) rank $\leq d$ [10]. There is no representation theorem for the maps which are P but not CP. Transpose map t is an example of P map which is not 2-P (hence not CP). Clearly, such P but not CP maps are the ones whose extensions, as given above, can detect entangled states. Woronowicz [12] showed that, when $(\dim(\mathcal{H}), \dim(\mathcal{K}))$ is one of the followings,- $(2,2)$, $(2,3)$, $(3,2)$; any P map $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is decomposable i.e. can be written as a sum of a CP and a transposed CP map. This shows that any state in $\mathbb{C}^2 \otimes \mathbb{C}^2$, $\mathbb{C}^2 \otimes \mathbb{C}^3$ (and $\mathbb{C}^3 \otimes \mathbb{C}^2$) is separable if and only if it is positive under partial transpose (PPT). This is not the case in higher dimensions. In all other dimensions there are PPT entangled states. These states naturally can not be detected by (extensions of) decomposable maps. Apart from the above mentioned cases, there exists indecomposable positive maps in all other dimensions.

3 Results

We highlight the main points of our result. The references are given below to the technical details. Most of the terms are explained in the section 2. For all others, we follow the terminologies of [10].

- Given any P map (not CP and indecomposable) $\varphi : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$, and any n-super-positive map $h : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$, the compositions $\varphi_h = \varphi \circ h$ and $\varphi^h = h \circ \varphi$ are P (not CP and indecomposable). Further if φ is extremal (i.e. can not be written as a convex combination of positive maps), and $h(x) = Ad_v(x) = V^\dagger x V$, then φ_h and φ^h are also extremal. This gives as an orbit of positive maps, as well as extremal orbit of positive maps. If φ and h are unital, so are the compositions.[2] For simplicity, let us assume that such super-positive maps are contractions (i.e. $V^\dagger V \leq I$, where I is the identity matrix).
- V is invertible, hence it can be written as $V = U_1 D U_2$ where U_1, U_2 are unitary operators and D is a diagonal matrix. Notice that local unitaries do not change the Bell measure of a state. Hence the measure is changed by application of the diagonal matrix D. This gives a potential connection of quantum filtration. Indeed, using an example we can show that when by using filtration we can increase the Bell measure of the state and then the state is detected by a given positive map.

Some of the results mentioned above are taken from the following paper

1. R Sengupta and Arvind. Extremal extensions of entanglement witnesses and their connections with UPB. arXiv:1211.3294 [quant-ph].

[2] h unital implies V is unitary. Hence we get unitary orbits.

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