Solution for the discrimination of three qubit-mixed quantum states

Donghoon Ha[1] * Younghun Kwon[1] †

[1] *Department of Applied Physics, Hanyang University, Ansan, Kyunggi-Do, 425-791, South Korea*

Abstract. In this article, by treating minimum error state discrimination as a complementarity problem, we obtain the geometric optimality conditions. These can be used as the necessary and sufficient conditions to determine whether every optimal measurement operator can be nonzero. Using these conditions and an inductive approach, we demonstrate a geometric method and the intrinsic polytope for N-qubit mixed-state discrimination. When the intrinsic polytope becomes a point, a line segment, or a triangle, the guessing probability, the necessary and sufficient condition for the exact solution, and the optimal measurement are analytically obtained. We apply this result to the problem of discrimination to arbitrary three-qubit mixed states with given a priori probabilities and obtain the complete analytic solution to the guessing probability and optimal measurement.

Keywords: quantum state discrimination, qubit, minimum-error, guessing probability, geometric optimality conditions

The goal of quantum state discrimination is to discriminate the quantum states of a given set as well as possible. In fact, in classical physics, every state can be orthogonal to each other and therefore perfectly distinguished [1]. However, in quantum physics, a state cannot be perfectly discriminated because of the existence of nonorthogonal states [2, 3, 4]. Quantum state discrimination [5] is classified into minimum error discrimination, originally introduced by Helstrom [2],unambiguous discrimination [6, 7, 8], and maximum confidence discrimination [9]. The purpose of minimum error strategy is to find the optimal measurement and the minimum error probability (or guessing probability) for arbitrary N qudit-mixed quantum states with arbitrary priori pobabilities. In the case of N = 2, regardless of the dimension, the Helstrom bound [2] gives an analytic solution to the problem. In the N = 3 case, the analytic solution for the pure qubit states is provided by [10, 11]. In [12] the analytic solution for the mixed qubit states is considered without the necessary and sufficient conditions for a solution. In other words, full understanding for the discrimination of 3 qubit-mixed quantum states has not yet been provided.

The von Neumann measurement [13] is used for optimal measurement for linearly independent quantum states . However, if the given quantum states are linearly dependent, the von Neumann measurement may not be optimal. Therefore, the Positive-Operator-Valued-Measure (POVM) should be used for arbitrary quantum states. From the point where POVM can be used as the measurement and the probability to guess the quantum states correctly becomes convex, the minimum error discrimination problem may be solved by convex optimization[14]. Other efforts to solve it have been made using the dual problem [15] or complementarity problem [16]. By applying qubit state geometry to the optimality conditions for the measurement operators and complementary states, Bae [17] obtained a geometric method to find the guessing probability and the optimal measurement for some special cases. However, they did not include the case where the optimal measurement cannot be POVM when every element is nonzero. In this paper, we show that the case where the optimal measurement cannot be POVM and where every element is nonzero can be understood through the existence of parameters satisfying the geometric optimality conditions [16]. We also clarify the meaning of these geometric conditions. Through the conditions and an inductive approach, we propose a method to discriminate arbitrary N qubit-mixed quantum states with arbitrary a priori probabilities. In this method, we define the intrinsic polytope for discrimination problems. When the polytope becomes a point, line segment, or triangle, we find the guessing probability, the necessary and sufficient condition for the exact solution, and the optimal measurement analytically. By the number of the extreme points for the intrinsic polytope and the geometric optimality conditions, we can provide a complete analysis for discrimination of the 3 qubit-mixed state. We also obtain its guessing probability and optimal measurement.

First of all, let us briefly explain the notations. Let q_i and $\rho_i (i = 1, \cdots, N)$ be the priori probability and $d \times d$ density matrix, where d and N denote the dimension and number of states to be discriminated. Hereafter, q_i is ordered by $q_i \geq q_{i+1}$. When $\{M_i\}_{i=1}^N$ is used for measurement to $\{q_i, \rho_i\}_{i=1}^N$, the probability to guess the quantum states correctly becomes $P_{\text{corr}} = \sum_{i=1}^N q_i tr \rho_i M_i$. The goal of the minimum error state discrimination is to obtain the maximum of P_{corr}, called *the guessing probability* P_{guess}, using POVM.

Here is our first lemma and corollary[18].

Lemma 1 (geometric KKT conditions) *The fact that every optimal POVM element can be nonzero is equivalent to the fact that* $\{r_i, \vec{w}_i\}_{i=1}^N$ *satisfying the geometric KKT conditions exists.*

Corollary 2 *If the number of the extreme points to* $P\{q_i, \rho_i\}_{i=1}^N$ *is one, every optimal POVM element except* M_1 *is zero, and the guessing probability is* q_1.

Furthermore we can obtain another corollary.

*merit.of.east@gmail.com

†yyhkwon@hanyang.ac.kr

Corollary 3 *If the number of the extreme points to* $P\{q_i, \rho_i\}_{i=1}^N$ *is two, the guessing probability becomes*

$$P_{\text{guess}} = \max_{i \neq j} \frac{1}{2}\left(q_i + q_j + \|q_i\rho_i - q_j\rho_j\|_1\right). \qquad (1)$$

When a *and* $b(> a)$ *are the indices giving the optimal value, if* $\|q_a\vec{v}_a - q_b\vec{v}_b\|_2 < q_a - q_b$*, every optimal POVM element except* M_1 *is zero. However, if* $\|q_a\vec{v}_a - q_b\vec{v}_b\|_2 \geq q_a - q_b$*, the optimal POVM elements are given as*

$$\begin{aligned} M_a &= \frac{1}{2}\left[I_2 + \left(\frac{q_a\vec{v}_a - q_b\vec{v}_b}{\|q_a\vec{v}_a - q_b\vec{v}_b\|_2}\right)\cdot\vec{\sigma}\right], \\ M_b &= \frac{1}{2}\left[I_2 + \left(\frac{q_b\vec{v}_b - q_a\vec{v}_a}{\|q_a\vec{v}_a - q_b\vec{v}_b\|_2}\right)\cdot\vec{\sigma}\right], \\ M_i &= 0 \quad \forall i \neq a, b. \end{aligned} \qquad (2)$$

From this we can have the following lemma.

Lemma 4 (three quantum states discrimination) *When arbitrary three quantum states* $\{q_i, \rho_i\}_{i=1}^3$ *are given with given priori probabilities, the guessing probability can be classified into the following three cases: (i) When the number of the extreme points to* $P\{q_i, \rho_i\}_{i=1}^3$ *is one, the guessing probability becomes* q_1 *by the corollary 2. (ii) When the number of the extreme points is two or three and the following conditions*

$$\begin{aligned} &\text{(i)} \quad l_1 > e_1,\ l_2 > e_2, \\ &\text{(ii)} \quad \frac{l_1\cos\theta_1 + e_1}{l_1 + e_1} < \frac{l_1 - e_1}{l_2 - e_2},\ \frac{l_2\cos\theta_1 + e_2}{l_2 + e_2} < \frac{l_2 - e_2}{l_1 - e_1}, \\ &\text{(iii)} \quad \frac{l_1^2 - e_1^2}{2(l_1\cos\chi + e_1)} < \frac{l_1\sin\theta_2}{\sin(\chi + \theta_2)}, \end{aligned} \qquad (3)$$

cannot be satisfied, the guessing probability can be found by the corollary 3. (iii) When the number of the extreme points is three and the condition of Eq.(3) is satisfied, the guessing probability can be given by

$$P_{\text{guess}} = q_1 + \frac{l_1^2 - e_1^2}{2(l_1\cos\chi + e_1)}. \qquad (4)$$

Acknowledgment

We would like to thank Dr. Bae Joonwoo for reading the paper and commenting. This work is supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (KRF2011-0027142 and KRF2012-0008086).

References

[1] A. Chefles, Contemp. Phys. **41**, 401 (2000)

[2] C. W. Helstrom, Quantum Detection and Estimation Theory (Academic, 1976).

[3] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (North-Holland 1979).

[4] S. M. Barnett, S. Croke, Adv. Opt. Photon. **1**, 238-278 (2009).

[5] J. A. Bergou, J. Mod. Opt. **57**, 3, 160-180 (2010).

[6] I. D. Ivanovic, Phys. Lett. A **123**, 257-259 (1987).

[7] D. Dieks, Phys. Lett. A **126**, 303-306 (1988).

[8] A. Peres, Phys. Lett. A **128**, 19-19 (1988).

[9] S. Croke, E. Andersson, S.M. Barnett, C.R. Gilson, J. Jeffers, Phys. Rev. Lett. **96**, 070401 (2006)

[10] K. Hunter, arXiv:quant-ph/0410228v1.

[11] B. F. Samsonov, Phys. Rev. A **80**, 052305 (2009).

[12] M. A. Jafarizadeh, Y. Mazhari, M. Aali, Quant. Info. Proc. 10, 155 (2011)

[13] Y. C. Eldar, Phys. Rev. A **68**, 052303 (2003).

[14] S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press (2004).

[15] M. E. Deconinck, B. M. Terhal, Phys. Rev. A **81**, 062304 (2010).

[16] J. Bae, W.-Y. Hwang, Phys. Rev. A **87**, 012334 (2013).

[17] J. Bae, arXiv:1210.2845.

[18] D. Ha, Y. Kwon, Phys. Rev. A **87**, 062302 (2013).