Maximal Average Fidelity in Quantum Teleportation of Single Qubit Mixed Information State by Using Two Qubits X-State as Resource

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Abstract: We considered standard quantum teleportation protocol of mixed (impure) single qubit information state where a general bipartite X-state is used as entanglement resource. We obtained an expression for maximal average fidelity (the fidelity averaged over all input states and maximized over general unitary operations done by receiver) in each case of sender's four Bell States Measurement results. It is also shown that, in a special case, this expression for fidelity is same as that given by R. Horodecki et.el. [Phys. Lett. A 222 (1996) 21].

Keywords: *X-states, QuantumTteleportation, Maximal Average Fidelity.*

Quantum entangled states are the key resource in the rapidly expanding field of quantum information science, with remarkable prospective applications such as quantum computation, quantum teleportation, super dense coding, quantum cryptographic schemes, entanglement swapping, and remote states preparation. Quantum teleportation, which describes the transmission of an unknown quantum state from a sender to a spatially distant receiver using quantum and classical channel, is one of the most astonishing features in quantum information and communication. In 1993, Bennett et al. [1] investigated that it was possible to transmit one-qubit state using Einstein–Podolsky–Rosen (EPR) state by sending two classical bits of information.

A pure separable state is represented by the product of states for both systems. The fidelity of the teleportation dependens on both the entangled state resource and the particular LOCC protocol [2-3]. H. Prakash and V. Verma [4] have shown that perfect QT is possible only when maximally entangled states are used as quantum channel. The maximal fidelity for separable states is indeed given by F=1/2. A natural question arises concerning teleportation, whether all states which violate the Bell-CHSH inequalities are suitable for teleportation. Horodecki *et al* [3] showed that any mixed two spin-1/2 state which violates the Bell-CHSH inequalities is suitable for teleportation of pure state information.

We consider the standard quantum teleportation protocol of mixed (impure) single qubit information state ($\rho_I = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 10| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|$) by using a general bipartite X-state as entanglement resource. Here $Tr(\rho_I) = \rho_{00} + \rho_{11} = 1$ and $|\rho_{01}|^2 \le \rho_{00}\rho_{11}$. A general two qubits X-state [5], which in the most general cases are mixed, is given by the density operator

$$\rho = A|00\rangle\langle 00| + B|01\rangle\langle 01| + C|10\rangle\langle 10| + D|11\rangle\langle 11| + F|00\rangle\langle 11| + G|01\rangle\langle 10| + G^*|10\rangle\langle 01| + F^*|11\rangle\langle 00|,$$

where A, B, C, D are real. Unit trace and positivity conditions give $A+B+C+D=1$, $|F| \le \sqrt{AD}$ and $|G| \le \sqrt{BC}$. X-states are entangled if and only if either $\sqrt{BC} \le |F|$ or $\sqrt{AD} \le |G|$. X states contain many important states like the four maximally entangled Bell states, the maximally mixed state, mixtures of maximally entangled and maximally mixed states like Werner states. For purity either $A = D = F = 0$ and $|G| = \sqrt{BC}$ or $B = C = G = 0$ and $|F| = \sqrt{AD}$. We address the following basic question: for a mixed X-state of two qubits, what is the maximal teleportation fidelity that can be obtained when this state is used as quantum channel for the teleportation of single qubit mixed information state. We find that the maximal average fidelity (the fidelity averaged over all input states and maximized over general unitary operations done by receiver) in each case of sender's four Bell States Measurement (BSM) results is given as

$$F_{av.max.} = \frac{1}{2} + \frac{1}{2}\gamma^2(|F| + |G|)I_3 + \frac{1}{2}\left|\frac{1}{2}\{(A-B)(1-x) + (D-C)(1+x)\}I + 2\gamma^2(|F| - |G|)I_3\right|$$

where $I = \frac{4}{x^2}[\frac{1}{2x}\log\left\{\frac{1+x}{1-x}\right\} - 1]$, $I_3 = 1 - \frac{1}{4}(1-x^2)I$, $x = (A+B) - (C+D)$, $0 \le x < 1$ and γ denotes purity of mixed information state defined as $|\rho_{01}|/\{\rho_{00}\rho_{11}\}^{1/2}$, giving $1 \le \gamma \le 0$. For pure information state $\gamma = 1$. The above expression of fidelity can also be written as

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$$F_{av.max} = \frac{1}{2} + \frac{1}{2}\gamma^2(|F| + |G|)I_3 + \frac{1}{2}\left|(AD - BC)I + \gamma^2(|F| - |G|)I_3\right| \ .$$

In a special case, when x = 0, the above expression of fidelity becomes

$$F_{av.max.} = \frac{1}{2} + \frac{1}{3}\gamma^2(|F| + |G|) + \frac{1}{6}\left|(A - B - C + D) + 2\gamma^2(|F| - |G|)\right|$$

For pure information state (γ = 1), expression for maximal average fidelity reduces to Horodecki et. al. [3] $F_{av.max} \le \frac{1}{2} + \frac{1}{6}N(\rho)$, where $N(\rho) = Tr[\sqrt{T^\dagger T}]$ and T is a 3×3 positive matrix with elements $t_{mn} = Tr[\rho\sigma_n \otimes \sigma_m]$ and σ's are usual Pauli spin operators. $N(\rho)$ can be expressed as sum of positive square roots of eigen values λ_i (i = 1, 2, 3) of the matrix $T^\dagger T$. For X-state the value of $N(\rho)$ is given by

$$N(\rho) = 2(|F| + |G|) + 2\left|(|F| - |G|)\right| + |A - B - C + D|.$$

Horodecki et al [3] also showed that any mixed two spin-1/2 state which violates Bell-CHSH inequality is useful for teleportation of pure state information. Recently Ming [5] repeated this problem using X-state and showed that any X-states which violate the Bell-CHSH inequality can also be used for non-classical teleportation if receiver can only perform the identity or Pauli rotation operations. We however see that this result for quantum channel cannot be extended to mixed state information. If we write $A = \cos^2\delta\cos^2\eta$, $B = \sin^2\delta\cos^2\eta$, $C = \sin^2\delta\sin^2\eta$, $D = \cos^2\delta\sin^2\eta$, $|F| = \frac{1}{2}\cos^2\delta\sin 2\eta$ and $|G| = \frac{1}{2}\sin^2\delta\sin 2\eta$ with $0 \le \delta \le \frac{\pi}{2}$ and $0 < \eta < \frac{\pi}{2}$. In this case, the real valued function $M(\rho)$ [3,5] defined as M(ρ) = max.{$\lambda_1+\lambda_2$, $\lambda_2+\lambda_3$} becomes

$$M(\rho) = \max.\{(1+\cos^2 2\delta)\sin^2 2\eta,\ \cos^2 2\delta + \sin^2 2\eta\}.$$

Also, we get $x = \cos 2\eta$ and the expression of the fidelity becomes

$$F_{av.max} = \frac{1}{2} + \frac{1}{4}\gamma^2\sin 2\eta\ I_3 + \frac{1}{8}\left|\cos 2\delta\sin 2\eta\right|\left|\sin 2\eta . I + 2\gamma^2 I_3\right|.$$

The Bell-CHSH inequality is violated [3, 5] if and only if M(ρ) > 1. At $\eta = \frac{\pi}{4}$, $M(\rho) = 1 + \cos^2 2\delta \ge 1$, x = 0 and $F_{av.max} = \frac{1}{2} + \frac{1}{6}\gamma^2 + \frac{1}{6}(1+\gamma^2)|\cos 2\delta|$. The variation of this fidelity is shown in Fig.(1). For γ = 1 (pure information) and $\eta = \frac{\pi}{4}$, $F_{av.max} = \frac{2}{3} + \frac{1}{3}|\cos 2\delta| \ge \frac{2}{3}$ and the variation of this fidelity is shown in Fig.(2). Since at $\eta = \frac{\pi}{4}$, $M(\rho) > 1$ provided $\delta \ne 0$ or $\frac{\pi}{2}$ and therefore Bell-CHSH inequality is violated. The Fig.(2) shows that in case of pure information state, all the X-states that violate Bell-CHSH inequality are suitable for non-classical teleportation. But from Fig.(1), it is clear that when information is mixed state, then, all the X-states which violates Bell-CHSH inequality are not suitable for non-classical teleportation. If $\delta = 0$ or $\frac{\pi}{2}$ further, X-state becomes Bell State. For this particular very special case, the non-classical teleportation occurs for any (pure or mixed) information state.

Fig. (1): For η = π/4

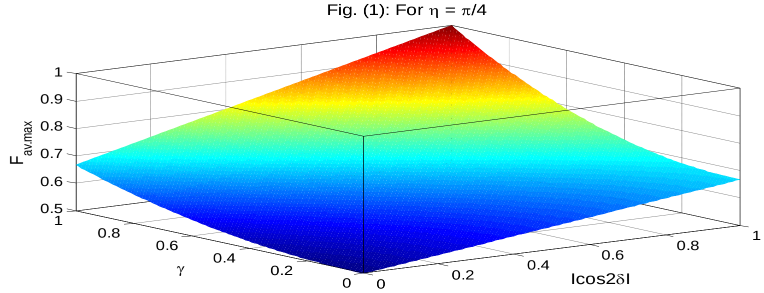
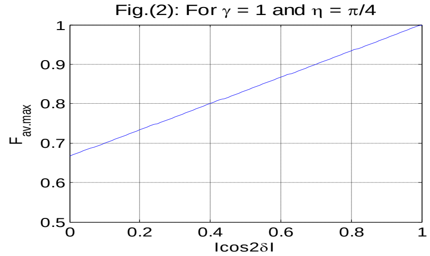


Fig.(2): For γ = 1 and η = π/4



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