Is 2-EPP good at low rates for a phase-damping channel ?

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Abstract. Recently, we demonstrated that a 2-EPP is superior to a 1-EPP if fifteen entangled states are initially shared via a phase-damping channel. We seek to prove that 2-EPP is superior to 1-EPPs when there is only one kind of error. In the present paper, we show property of exponential bounds on fidelity of EPPs, which correspond to bounds on error exponent in usual information transmission. As a result, an exponential bound for the fidelity of a particular 2-EPP is always higher than that of 1-EPPs, at least at low rates if the channel is assumed to be phase damped.

Keywords: entanglement purification, quantum error correction, reliability function, exponential bound

1 Introduction

Entanglement is a known useful resource in quantum information processing; in particular, sharing a maximally entangled state is important. To share such states via a noisy quantum channel, entanglement purification protocols (EPPs) [1] are essential. There are two classes of EPPs: 1-EPPs, which uses one-way classical communication, and 2-EPPs, which uses two-way communication. Although 2-EPPs are superior to 1-EPPs in general settings, it seems that, because the upper limits for the yield [1] for both classes coincide, 1-EPPs are sufficient to purify entangled states degraded in a channel with error of one kind (e.g., phase-damping channel). Recently, we showed that a 2-EPP is superior to a 1-EPP when fifteen entangled states are initially shared via a phase-damping channel [2]. This result implies that a 2-EPP is superior to 1-EPPs when the number of initially shared entangled states is finite. Nevertheless, no 2-EPP outperforms the best 1-EPP in the infinite limit. In information theory terms, a 2-EPP is superior to 1-EPPs at much lower rates than the 'capacity' even if there is only one kind of error in the channel. In classical and quantum information theories, several examples are known that show superiority at low rates whereas capacities are not improved [3, 4]. For example, in classical information theory, although it is well-known that channel capacities with and without feedback are identical, feedback is effective if the rate is lower than the capacity [3]. In quantum information theory, the usefulness of squeezed states was shown at low rates [4] despite the fact that the channel capacity has been attained by coherent states. Whether classical or quantum, bounds on the reliability function or error exponent are employed to exploit advantage at low rates.

In the present paper, we derive exponential bounds on the $1-F$ of EPPs, which correspond to bounds on the error exponent in normal information transmissions. As a result, an exponential bound on the fidelity of a particular 2-EPP is shown to be higher than a bound for the fidelity of 1-EPPs at least at low rates when the channel is assumed to be phase damped. The latter result clearly shows that a 2-EPP is superior to 1-EPPs and using a 2-EPP saves on the number of initial shared entangled states even if there is only one kind of error.

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2 Basic notions

In this section, we explain the setting of the problem and describe channels, EPPs, and their evaluation. In our study, we considered the following problem. Alice prepares n Bell states $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$, and she sends half of each state to Bob over a noisy quantum channel. Then, Alice and Bob each apply an EPP to their states; if they keep their pairs, they share k entangled states.

2.1 Noisy quantum channels

First, we define a qubit channel as given in Ref.[5]. Let σ be an input qubit state of the channel. The output state of the channel is then described as

$$\mathcal{A}(\sigma) = P((0,0))\sigma + P((1,0))X\sigma X^\dagger + P((0,1))Z\sigma Z^\dagger + P((1,1))(XZ)\sigma(XZ)^\dagger, \quad (1)$$

where $u = (i,j) \in \{0,1\}^2$, $P(u)$ is a probability satisfying $0 \le P(u) \le 1$ and $\sum_u P(u) = 1$. The behavior of a phase-damping channel corresponds to the probability of having properties $P((0,0)) = 1-p, P((1,0)) = p, P((0,1)) = P((1,0)) = 0$ in Eq.(1).

2.2 Entanglement purification protocols

EPPs are protocols to distil n pairs of mixed entangled states $\rho^{\otimes n}$ into k pairs of (near) maximally entangled states ρ_{out} [1]. Two classes of EPPs, 1-EPP and 2-EPP, exist. Recall that a 1-EPP was shown to be equivalent to a quantum error correcting code (QECC) [1]. That is, for any (n,k) QECC, one can construct a 1-EPP.

2.3 Evaluation of EPPs

Two evaluation factors for EPPs are known, fidelity and purification rate. The fidelity F is the overlap between k Bell states $|\Phi^+\rangle^{\otimes k}$ and the remaining entangled states $\rho_{\rm out}$ after the EPP application; F is formally defined as $F = {}^{\otimes k}\langle\Phi^+|\rho_{\rm out}|\Phi^+\rangle^{\otimes k}$, where $0 \le F \le 1$. The purification rate $R_{\rm P}$ for shared entangled states is defined as $R_{\rm P} = \frac{k}{n}P_{\rm S}$, where $P_{\rm S}$ is the success probability of the EPP. We refer to the limit $D = \lim_{n\to\infty} R_{\rm P}$ under $F \to 1$ as the *yield* for the EPPs. From the equivalence of a 1-EPP with a QECC, the upper limit of the yield of 1-EPPs is equal to the quantum capacity.

3 Exponential bound on fidelity

In Ref.[5], an exponential bound on the fidelity for quantum information transmission was derived by applying the concept of the classical reliability function to quantum channels. According to [5], the minimum fidelity $F^\star_{n,k}$ with the best (n,k) QECC is lower bounded as follows:

$$F^\star_{n,k} \ge 1 - (n+1)^{2(2^2-1)} d^{-nE(R,P)}, \tag{2}$$

where

$$E(R,P) = \min_Q [D(Q||P) + |1 - H(Q) - R|^+], \tag{3}$$

R corresponds to the rate of the QECC, and we assume the qubit channel is characterized by P (of Eq.(1)). For more details, see Ref.[5]. We refer to $E(R,P)$ for this paper as the exponential lower bound on fidelity.

Because a 1-EPP is equivalent to a QECC, the bound $E(R,P)$ is applicable in an evaluation of 1-EPPs. Although tightness is an important factor to treat a *bound*, the bound is expected to be tight as far as a phase-damping channel is concerned because of its relationship to the random-coding bound in the classical theory. As for 2-EPPs, to derive a tight exponential bound on fidelity is not an easy task. Instead of deriving this bound, we concentrate on verifying the superiority of 2-EPPs to 1-EPPs. For this purpose, we follow the method used in Ref.[3], in which a tight bound on the error exponent without feedback is used, whereas the error exponent by a specific code is used for a feedback scheme.

4 Superiority of a 2-EPP

In this section, we show superiority of a 2-EPP. As mentioned in the previous section, we employ the exponential lower bound $E_2(R,P)$ for fidelity of a specific protocol as evaluation of a 2-EPP and employ the exponential lower bound $E(R,P)$ of fidelity of the best protocol as a means to evaluate 1-EPPs. For this evaluation, 2-EPPs with a variable number of initial shared entangled states might be required. However, following the method in Ref.[1], we consider a 2-EPP that is simple and finite, followed by an asymptotic 1-EPP such as one-way hashing. We use the recurrence method [1] as the simple 2-EPP. We then have a bound $E_2(R,P)$ by computing $E(R,P)$ of a 1-EPP whose input states are the outputs of the recurrence protocol.

We evaluate $E(R,P)$ and $E_2(R,P)$ for a channel assumed here to be phase-damped. Because the bound $E(R,P)$ is expected to be tight for a phase-damping channel, the result in this subsection provides a clear conclusion.

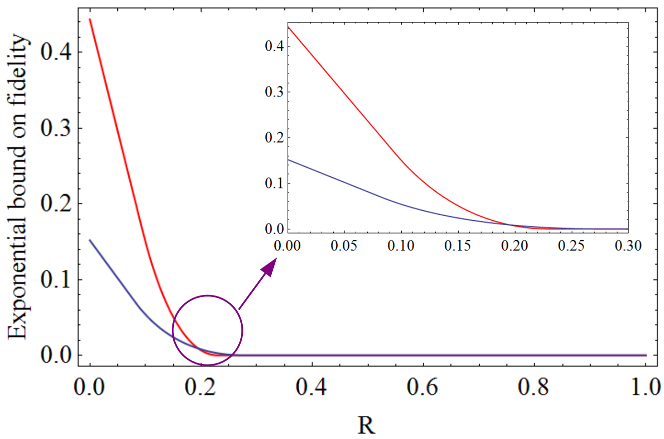

Figure 1: Exponential bound on fidelity for a phase-damping channel with $p = 0.2$.

Figure 1 presents plots of $E(R,P)$ (blue line) and $E_2(R,P)$ (red line) when $p = 0.2$. Here, p is defined in Sec.2.1. As the simple 2-EPP is performed before performing the asymptotic 1-EPP, the yield for the 2-EPP is smaller than that of the 1-EPP. However, we find that $E_2(R,P) > E(R,P)$ when $R \lesssim 0.2$ inferring that this specific 2-EPP is superior to 1-EPPs at low rates.

5 Conclusion

By comparing the exponential bound on $1-F$ for a specific 2-EPP to that of 1-EPPs, we see that the bound on the former is higher than that on the latter at low rates, even if the yield of the 2-EPP is less than that of the 1-EPPs. The result shows that a 2-EPP is superior to the 1-EPPs and use of a 2-EPP saves on the number of initial shared entangled states even if there is only one kind of error. We shall develop a tight exponential lower bound on $1-F$ for 2-EPPs in a future study.

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References

[1] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, "Mixed state entanglement and quantum error correction," Phys. Rev. **A54**, pp.3824-3851, (1996).

[2] Y. Yazaki, S. Usami, and T.S. Usuda, "Superiority of 2-EPPs to 1-EPPs with finite entangled resources," Proc. of ISITA2012, pp.206-210, (2012).

[3] J.P.M. Schalkwijk and T. Kailath, "A coding scheme for additive noise channels with feedback-Part I: No bandwidth constraint," IEEE Trans. on Inform. Theory **IT-12**, no.2, pp.172-182, (1966).

[4] M. Sohma and O. Hirota, "Squeezing is good at low information rates," Phys. Rev. **65**, 022319, (2002).

[5] M. Hamada, "Exponential lower bound on the highest fidelity achievable by quantum error-correcting codes," Phys. Rev. **A65**, 052305, (2002).