Maximization of Holevo information for entanglement-assisted classical communication using quasi-Bell states

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Abstract. Entangled states using certain nonorthogonal states called "quasi-Bell states" are important resources in quantum information systems. We consider entanglement-assisted classical communication using quasi-Bell states. In this case, simultaneous optimization of an *a priori* distribution and an encoding function is desired to compute the capacity. However, to optimize both quantities directly is computationally hard, and a reduction in the computational complexity is necessary. In the present paper, we apply a quantum version of the Arimoto-Blahut algorithms to compute the capacity. As a result, the simultaneous optimization of the *a priori* distribution and the encoding function is achieved.

Keywords: entanglement, quasi-Bell state, Arimoto-Blahut algorithms, Holevo information, capacity

1 Introduction

In quantum information systems, entanglement is viewed as an important resource. Entangled states using nonorthogonal states called "quasi-Bell states"[1], such as coherent states of light, have been shown to be capable of "perfect entanglement" and are expected to be robust against attenuations in quantum channels.

In this study, we consider the entanglement-assisted classical communication[2] using quasi-Bell states. Using this protocol, we can obtain two bits by transmitting a 1-qubit in the ideal qubit channel. In two previous studies, classical information transmission using quasi-Bell states was considered for an ideal channel[3] and a lossy channel[4].

In these papers, an approximate encoding was assumed that works for sufficiently large coherent amplitudes. Recently, we showed the capacity of a classical communication using degraded quasi-Bell states based on rigorously realizable encodings[5]. In Ref.[5], the capacity was computed by optimizing an *a priori* distribution, although a fixed encoding function was used. However, what is more desired is the simultaneous optimization of the *a priori* distribution and the encoding function. To optimize both quantities directly though is computationally hard, hence some reduction in computational complexity is necessary.

In the present paper, we apply a quantum version of the Arimoto-Blahut algorithm[6] to compute the capacity. As a result, simultaneous optimization of the *a priori* distribution and the encoding function is achieved.

2 Preliminary

2.1 Quantum Arimoto-Blahut algorithms[6]

The quantum Arimoto-Blahut algorithms are a well-known technique for computing the capacity of a quantum channel or the Holevo capacity[6]. We apply it in part to our problem.

Let $\Pi_n = \{(p_1, \cdots, p_n; \sigma_1, \cdots, \sigma_n)\}$ be the set of possible inputs. Here, $\{p_i\}$ is the *a priori* probability distribution and $\{\sigma_i\}$ is the set of input quantum states. Let $I(\pi)$ be the Holevo information or the quantum mutual information for an input $\pi = (p_i, \sigma_i) \in \Pi_n$. Then the Holevo capacity is defined as

$$C = \sup_{\pi \in \Pi_n} I(\pi). \tag{1}$$

In Ref.[6], a two-variable extension $J(\pi, \pi')$ of $I(\pi)$ was introduced, which satisfies

$$I(\pi) = J(\pi, \pi) = \max_{\pi' \in \Pi_n} J(\pi, \pi'). \tag{2}$$

Let the sequence $\{\pi^{(k)}\}_{k=1}^{\infty}$ be defined by

$$\pi^{(k+1)} = \operatorname*{argmax}_{\pi} J(\pi, \pi^{(k)}). \tag{3}$$

The recursion $\pi^{(r+1)}$ that a set of states is fixed is shown the following equations.

$$p_i^{(k+1)} = p_i^{(k)} \exp\left(\mathrm{Tr}\left[\mathcal{E}(\hat{\sigma_i}^{(k+1)}) \Phi(\hat{\sigma_i}^{(k)}, \hat{\rho}^{(k)}) \right] \right) / Z^{(k+1)} \tag{4}$$

where $Z^{(k+1)}$ is a normalizing constant, then

$$I(\pi^{(k)}) \leq J(\pi^{(k+1)}, \pi^{(k)}) \leq I(\pi^{(k+1)}). \tag{5}$$

We can thus recursively compute $I(\pi^{(k)})$; in the limit, $I(\pi^{(\infty)})$ is expected to be the Holevo capacity if the algorithm works well.

2.2 Quasi-Bell state by coherent states

Quasi-Bell states are based on nonorthogonal states[1]. A coherent state is denoted as $|\alpha\rangle$ and has amplitude α. We use one specific state $|\Psi_4\rangle = h_4(|0\rangle_A|0\rangle_B - |\alpha\rangle_A|\beta\rangle_B)$.

2.3 Encoding

An encoding function is a local operation for the mode A corresponding to a classical signal i. We assume that this encoding function is represented by the following unitary operator,

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$$
U_A^{(i)} = \begin{cases} R_1(\theta^{(i)}) = \begin{bmatrix} \cos(\theta^{(i)}) & -\sin(\theta^{(i)}) \\ \sin(\theta^{(i)}) & \cos(\theta^{(i)}) \end{bmatrix}, \\ \qquad\qquad\qquad (0 \leq \theta^{(i)} < 2\pi) \\ R_2(\theta^{(i)}) = \begin{bmatrix} \cos(\theta^{(i)}) & \sin(\theta^{(i)}) \\ \sin(\theta^{(i)}) & -\cos(\theta^{(i)}) \end{bmatrix}, \\ \qquad\qquad\qquad (2\pi \leq \theta^{(i)} < 4\pi). \end{cases} \tag{6}
$$

We can then define the following input quantum states:

$$
\hat{\sigma}_i^{(\mathrm{A}\otimes\mathrm{B})} = [U_{\mathrm{A}}^{(i)} \otimes I_{\mathrm{B}}]\hat{\sigma}^{(\mathrm{A}\otimes\mathrm{B})}[U_{\mathrm{A}}^{(i)} \otimes I_{\mathrm{B}}]^\dagger \tag{7}
$$

where $i = 1, 2, 3, 4$, $\hat{\sigma}^{(\mathrm{A}\otimes\mathrm{B})} = |\Psi_4\rangle\langle\Psi_4|$, and I_B is the identity operator.

3 Computing the capacity

We consider the simultaneous optimization of the *a priori* distribution and the encoding function. Here, we use the quantum Arimoto-Blahut algorithm only to optimize the probability distribution; the input quantum states are optimized using another algorithm. Because, the input quantum states in our problem are restricted to states obtained by local operations, the quantum Arimoto-Blahut algorithm can not be directly applied in optimizing these states. However, because these states can be optimized using a single-parameter family of transformations [5], their optimization is computationally an easy problem and any algorithm is acceptable. Therefore, we assume that a recursion of input states $\sigma^{(k+1)}$ can be found by searching $\theta^{(i)}$. Then, we assume that the recursion $\sigma^{(k+1)}$ is given by $\sigma^{(k+1)} = \arg\max_{0\leq\theta\leq 4\pi} \mathrm{Tr}[\mathcal{E}(\hat{\sigma}_i(\theta^{(i)}))\Phi(\hat{\sigma}_i^{(r)}, \hat{\rho}^{(r)})]$. Figure 1 shows the Holevo information $J(\pi^{(r)}, \pi^{(r)})$ in the recursion process $\pi^{(r)} \to \pi^{(r+1)}$; Figure 2 shows the variation of the Holevo information $J(\pi^{(r+1)}, \pi^{(r+1)}) - J(\pi^{(r)}, \pi^{(r)})$. We see that this variation approaches 0 when $r \gtrsim 10$. Because the limit value might be a local maximum, we executed the algorithm many times by changing the initial input distributions. Figure 1 is an example. Although the values for $J(\pi^{(r)}, \pi^{(r)})$ are different depending on the initial inputs for small r and are not monotonically nondecreasing, these converge to the same value. Therefore, the obtained value is a global maximum and full optimization is achieved.

4 Conclusion

We applied the quantum Arimoto-Blahut algorithm to the computation of the Holevo capacity for entanglement-assisted classical communication when the quasi-Bell states are used as shared initial entanglements.

The capacity was computed quickly and with high precision. As a typical example, when compared with a random search task, which was used in our previous study, the precision was 10000 times higher and the computation time was more than 100 times faster.

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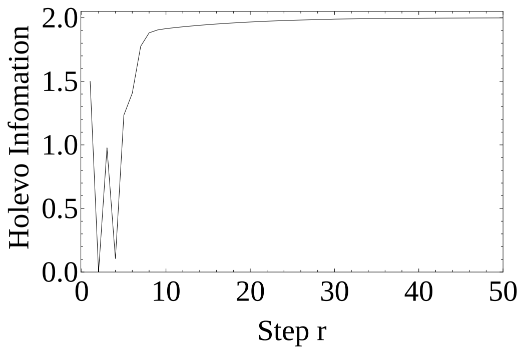

Figure 1: Holevo information $J(\pi^{(r)}, \pi^{(r)})$ in the recursion process $\pi^{(r)} \to \pi^{(r+1)}$

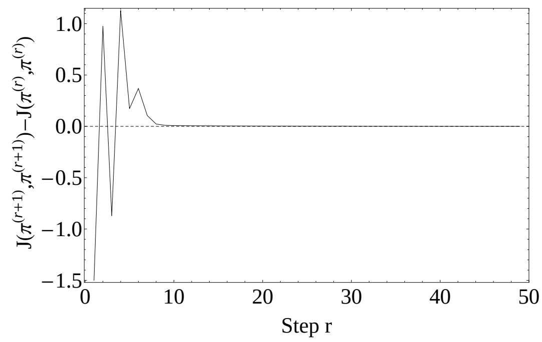

Figure 2: $J(\pi^{(r+1)}, \pi^{(r+1)}) - J(\pi^{(r)}, \pi^{(r)})$

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