Quantum discord plays no distinguished role in characterization of complete positivity : Robustness of the traditional scheme

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Abstract. The traditional scheme for realizing open-system quantum dynamics takes the initial state of the system-bath composite as a simple product. Currently, however, the issue of system-bath initial correlations possibly affecting the reduced dynamics of the system has been attracting considerable interest. The influential work of Shabani and Lidar [PRL **102**, 100402 (2009)] famously related this issue to quantum discord, a concept which has in recent years occupied the centre-stage of quantum information theory and has led to several interesting results. They have claimed that reduced dynamics is completely positive if and only if the initial system-bath correlations have vanishing quantum discord. Here we show that there is, within the Shabani-Lidar framework, no scope for any distinguished role for quantum discord in respect of complete positivity of reduced dynamics. Since most applications of quantum theory to real systems rests on the traditional scheme, its robustness demonstrated here could be of far-reaching significance.

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Every physical system is in interaction with its environment, *the bath*, to a smaller or larger degree of strength. As a consequence, unitary Schrödinger evolutions of the composite, the system plus the bath, manifests as dissipative non-unitary evolutions when restricted to the system of interest. The folklore scheme for realizing such open system dynamics is to elevate the system states ρ_S to the (tensor) products $\rho_S \otimes \rho_B^{\rm fid}$, for a *fixed* fiducial bath state $\rho_B^{\rm fid}$, then to evolve these uncorrelated system-bath states under a joint unitary $U_{SB}(t)$, and finally to trace out the bath degrees of freedom to obtain the evolved states $\rho_S(t)$ of the system:

$$\begin{aligned}\rho_S \to \rho_S \otimes \rho_B^{\rm fid} \to U_{SB}(t)\, \rho_S \otimes \rho_B^{\rm fid}\, U_{SB}(t)^\dagger \\ \to \rho_S(t) = \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\, \rho_\mathrm{S} \otimes \rho_\mathrm{B}^{\rm fid}\, \mathrm{U_{SB}(t)}^\dagger\right]. \quad (1)\end{aligned}$$

The resulting quantum dynamical process (QDP) $\rho_S \to \rho_S(t)$, parametrized by $\rho_B^{\rm fid}$ and $U_{SB}(t)$, is provably completely positive (CP).

While every CP map can be thus realized with uncorrelated initial states of the composite, a suspicion that more general realizations of CP maps could be possible has always been lurking beneath the surface. A specific, carefully detailed, and precise formulation of the issue of initial system-bath correlations possibly influencing the reduced dynamics was presented not long ago by Shabani and Lidar (SL) [1]. In this formulation, the distinguished bath state $\rho_B^{\rm fid}$ is replaced by a collection of (possibly correlated) system-bath initial states $\Omega^{SB} \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$, where $\mathcal{H}_S$, $\mathcal{H}_B$ are the Hilbert spaces of the system and bath, the dimensions being d_S, d_B respectively. The dynamics gets defined through a joint unitary $U_{SB}(t)$:

$$\begin{aligned}\rho_{SB}(0) \to \rho_{SB}(t) = U_{SB}(t)\, \rho_{SB}(0)\, U_{SB}(t)^\dagger, \\ \forall\ \rho_{SB}(0) \in \Omega^{SB}. \quad (2)\end{aligned}$$

This composite dynamics induces on the system the QDP

$$\rho_S(0) \to \rho_S(t), \quad (3)$$

with $\rho_S(0)$ and $\rho_S(t)$ defined through this natural imaging from Ω^{SB} to the system state space Λ_S:

$$\rho_S(0) = \mathrm{Tr_B}\, \rho_{\rm SB}(0),\ \ \rho_\mathrm{S}(\mathrm{t}) = \mathrm{Tr_B}\rho_{\rm SB}(\mathrm{t}).$$

It is evident that the folklore scheme, the *Stinespring dilation*, obtains as the special case $\Omega^{SB} = \{\,\rho_S \otimes \rho_B^{\rm fid} \,|\, \rho_B^{\rm fid} = \text{fixed}\,\}$.

This generalized formulation of QDP allows SL to transcribe the fundamental issue to this question: What are the necessary and sufficient conditions on the collection Ω^{SB} so that the induced QDP $\rho_S(0) \to \rho_S(t)$ in Eq. (3) is guaranteed to be CP *for all* joint unitaries $U_{SB}(t)$? Motivated by the work of Rodriguez-Rosario et al. [2], and indeed highlighting it as *'a recent breakthrough'*, SL advance the following resolution to this issue:

Theorem 1 (SL)*: The QDP in Eq.* (3) *is CP for all joint unitaries $U_{SB}(t)$ if and only if the quantum discord vanishes for all $\rho_{SB} \in \Omega_{SB}$, i.e., if and only if the initial system-bath correlations are purely classical.*

Obviously, in order that the QDP in Eq. (3) be *well defined* in the first place, the set Ω^{SB} should necessarily satisfy the following two properties:

Property 1: No state $\rho_S(0)$ can have two (or more) pre-images in Ω^{SB}. If a state $\rho_S(0)$ has two pre-images $\rho_{SB}(0) \neq \rho_{SB}^{'}(0)$ with $\mathrm{Tr_B}\rho_{\rm SB}(0) = \mathrm{Tr_B}\rho_{\rm SB}^{'}(0)$, there exists certainly some $U_{SB}(t)$ such that $\rho_S(t) = \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\rho_{\rm SB}(0)\mathrm{U_{SB}(t)}^\dagger\right] \neq \rho_S'(\mathrm{t}) = \mathrm{Tr_B}\left[\mathrm{U_{SB}(t)}\rho_{\rm SB}'(0)\mathrm{U_{SB}(t)}^\dagger\right]$, making one and the same system state $\rho_S(0)$ evolve into two distinct states $\rho_S(t)$ and $\rho_S^{'}(t)$, and thus rendering the QDP in Eq. (3) one-to-many, and hence ill-defined.

Property 2: While every system state $\rho_S(0)$ need not have a pre-image *actually enumerated* in Ω^{SB}, the set of $\rho_S(0)$'s having pre-image should be *'large enough that the QDP in Eq.* (3) *can be extended by linearity to all states of the system'* [2]. If Ω^{SB} fails this property, then the very issue of CP would make no sense. For, in carrying out verification of CP property, the QDP would be required to act, as is well known, on the operator basis $\{|j\rangle\langle k|\}$ for $j,\, k\, =\, 1, 2, \cdots d_S$; i.e., on generic complex d_S-dimensional square matrices, and not just on positive or hermitian matrices alone. Since the basic issue on hand is to check if the QDP as a map on $\mathcal{B}(\mathcal{H}_S)$ is CP or not, it is essential that it be well defined (at least by linear extension) on the entire *complex* linear space $\mathcal{B}(\mathcal{H}_S)$.

Our entire analysis rests on these two innocent-looking and nearly-obvious minimal requirements on the set Ω_{SB}. We 'assume', to begin with, that every pure state $|\psi\rangle$ of the system has a pre-image in Ω^{SB}. It is evident that, for every pure state $|\psi\rangle$, the pre-image in Ω^{SB} has to necessarily assume the (uncorrelated) product form $|\psi\rangle\langle\psi| \otimes \rho_B$, ρ_B being a state of the bath which could possibly depend on the system state $|\psi\rangle$. While this is self-evident and is actually independent of SL, it may be viewed as a consequence of the necessary condition part of SL theorem.

Now, let $\{|\psi_k\rangle\}_{k=1}^{d_S}$ be an orthonormal basis in $\mathcal{H}_S$ and let $\{|\phi_\alpha\rangle\}_{\alpha=1}^{d_S}$ be another orthonormal basis related to the former through a complex Hadamard unitary matrix U. Recall that a unitary U is Hadamard if $|U_{k\alpha}| = 1/\sqrt{d_S}$, independent of k, α. For instance, the characters of the cyclic group of order d_S written out as a $d_S \times d_S$ matrix is Hadamard. The fact that the $\{|\psi_k\rangle\}$ basis and the $\{|\phi_\alpha\rangle\}$ basis are related by a Hadamard means that the magnitude of the inner product $\langle\psi_k|\phi_\alpha\rangle$ is independent of both k and α, and hence equals $1/\sqrt{d_S}$ uniformly. We may refer to such a pair as *relatively unbiased bases*.

Let $|\psi_k\rangle\langle\psi_k| \otimes O_k$ be the pre-image of $|\psi_k\rangle\langle\psi_k|$ and $|\phi_\alpha\rangle\langle\phi_\alpha| \otimes \widetilde{O}_\alpha$ be that of $|\phi_\alpha\rangle\langle\phi_\alpha|$, $k, \alpha = 1, 2, \cdots, d_S$. Possible dependence of the bath states O_k on $|\psi_k\rangle$ and $\widetilde{O}_\alpha$ on $|\phi_\alpha\rangle$ has not been ruled out as yet. Since the maximally mixed system state can be expressed in two equivalent ways as $(d_S)^{-1}\sum_k |\psi_k\rangle\langle\psi_k| = (d_S)^{-1}\sum_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$, *uniqueness* of its pre-image in Ω^{SB} (Property 1) demands

$$\sum_{k=1}^{d_S} |\psi_k\rangle\langle\psi_k| \otimes O_k = \sum_{\alpha=1}^{d_S} |\phi_\alpha\rangle\langle\phi_\alpha| \otimes \widetilde{O}_\alpha.$$

Taking projection of both sides on $|\psi_j\rangle\langle\psi_j|$, and using $|\langle\psi_j|\phi_\alpha\rangle|^2 = (d_S)^{-1}$, we have

$$O_j = \frac{1}{d_S}\sum_{\alpha=1}^{d_S} \widetilde{O}_\alpha, \quad j = 1, 2, \cdots, d_S,$$

while projection on $|\phi_\beta\rangle\langle\phi_\beta|$ leads to

$$\widetilde{O}_\beta = \frac{1}{d_S}\sum_{k=1}^{d_S} O_k, \quad \beta = 1, 2, \cdots, d_S.$$

These $2d_S$ constraints together imply that $O_j = \widetilde{O}_\beta$ uniformly for all j, β. Thus the pre-image of $|\psi_k\rangle\langle\psi_k|$ is $|\psi_k\rangle\langle\psi_k| \otimes \rho_B^{\rm fid}$ and that of $|\phi_\alpha\rangle\langle\phi_\alpha|$ is $|\phi_\alpha\rangle\langle\phi_\alpha| \otimes \rho_B^{\rm fid}$, for all k, α, for some *fixed bath state* $\rho_B^{\rm fid}$. And, perhaps more importantly, the pre-image of the maximally mixed state $(d_S)^{-1}1\!\!1$ necessarily equals the product $(d_S)^{-1}1\!\!1 \otimes \rho_B^{\rm fid}$.

Taking another pair of relatively unbiased bases $\{|\psi_k^{'}\rangle\}$, $\{|\phi_\alpha^{'}\rangle\}$ one similarly concludes that the pure states $|\psi_k^{'}\rangle\langle\psi_k^{'}|$, $|\phi_\alpha^{'}\rangle\langle\phi_\alpha^{'}|$ too have pre-images $|\psi_k^{'}\rangle\langle\psi_k^{'}| \otimes \rho_B^{\rm fid}$, $|\phi_\alpha^{'}\rangle\langle\phi_\alpha^{'}| \otimes \rho_B^{\rm fid}$ respectively, with the same fixed fiducial bath state $\rho_B^{\rm fid}$. This is so, since the maximally mixed state is *common* to both sets.

Considering in this manner enough number of pure states or projections $|\psi\rangle\langle\psi|$ sufficient to span—by linearity—the entire system state space Λ_S, and hence $\mathcal{B}(\mathcal{H}_S)$, one readily concludes that *every element* of Ω^{SB} *necessarily* needs to be of the product form $\rho_S(0) \otimes \rho_B^{\rm fid}$, for some *fixed* bath state $\rho_B^{\rm fid}$. But this is exactly the folklore or Stinespring realization of non-unitary dissipative dynamics given in Eq. (1), to surpass which was the primary goal of the SL scheme. We have thus proved our principal result :

Theorem 2 *: No initial correlations—even classical ones—are permissible within the SL scheme.*

While our proof above was under an assumption, it can be shown that the conclusion is valid even without this assumption [3]. It is worth emphasising that our *entire analysis* is based *only on* the two nearly-obvious and innocent-looking minimal requirements on Ω^{SB} motivated above.

Since the SL theorem has influenced an enormous number of authors, it is possible that some results of those authors need recalibration in the light of our result, particularly if those results made essential use of the sufficiency part of the SL theorem. The SL theorem has come to be regarded among the more important recent results of quantum information theory, and several authors have paraded it as one of the major achievements of quantum discord. Our analysis conclusively proves that characterization of complete positivity of QDP is not an item that can enter the resumé of quantum discord.

But there is another, possibly much deeper, implication of our finding. Our analysis-strictly staying within the SL framework has shown that this framework brings one exactly back to the folklore scheme itself, as if it were a fixed point. This is not at all a negative result: the fact that the folklore product-scheme survives attack under this well-defined, powerful, and fairly general framework of attack demonstrates its, perhaps unsuspected, robustness. 'Stinespring Regained', if one likes.

References

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