Is nonclassicality-breaking the same thing as entanglement-breaking?

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Abstract. Nonclassicality and entanglement are notions fundamental to quantum information processes involving continuous variable systems. That these two notions are intimately related has been intuitively appreciated for quite some time. An aspect of considerable interest is the behaviour of these attributes of a state under the action of a noisy channel. Inspired by the notion of entanglement-breaking channels, we define the concept of nonclassicality-breaking channels in a natural manner. We show that the notion of nonclassicality-breaking is essentially equivalent—in a clearly defined sense of the phrase 'essentially'—to the notion of entanglement-breaking, as far as bosonic Gaussian channels are concerned. This is notwithstanding the fact that the very notion of entanglement-breaking requires reference to a bipartite system, whereas the definition of nonclassicality-breaking makes no such reference. Our analysis rests on our classification of channels into nonclassicality-based, as against entanglement-based, types of canonical forms. Our result takes one's intuitive understanding of the close relationship between nonclassicality and entanglement a step closer.

Keywords: Nonclassicality, Entanglement

Two notions that have been particularly well explored in the context of quantum information of continuous variable states are *nonclassicality* and *entanglement*. The 'older' notion of entanglement has become one of renewed interest in recent decades for its central role and applications in (potential as well as demonstrated) quantum information processes, while the concept of nonclassicality which emerges directly from the *diagonal representation* had already been well explored in the quantum optical context, long before the emergence of the present quantum information era. While nonclassicality can be defined even for states of a single mode of radiation, the very notion of entanglement requires two or more parties. Nevertheless, it turns out that the two notions are not entirely independent of one another; they are rather intimately related. In fact, nonclassicality is a prerequisite for entanglement. Since a nonclassical bipartite state whose nonclassicality can be removed by local unitaries could not be entangled, one can assert, at least in some intuitive sense, that '*entanglement is nonlocal nonclassicality*'.

An important aspect in the study of nonclassicality and entanglement is in regard of their evolution under the action of a channel. A noisy channel acting on a state can degrade its nonclassical features. Similarly, bipartite entanglement can be degraded by channels acting locally on the constituent parties or modes. In fact, there are channels that render every bipartite state separable by acting on just one of the parties [1]. Such channels are said to be *entanglement-breaking*.

A class of channels that has been of particular interest in the continuous variable quantum information processing context is the family of Gaussian channels. These are physical processes that map Gaussian states into Gaussian states. A (centered) Gaussian state is completely specified by its variance matrix V, and under the action of Gaussian channel Γ specified by the pair of matrices (X, Y), $Y \geq 0$ we have $V \rightarrow V' = X^T V X + Y$.

In this work we address the following issue: *which Gaussian channels have the property that they rid every input state of its nonclassicality*? We recall that the density operator $\hat{\rho}$ representing any state of radiation field is '*diagonal*' in the coherent state '*basis*', and this happens because of the over-completeness property of the coherent state basis. An important notion that arises from the diagonal representation is the *classicality-nonclassicality divide*. Since coherent states are the most elementary of all quantum mechanical states exhibiting classical behavior, any state that can be written as a convex sum of these elementary classical states is deemed classical. Any state which cannot be so written as a convex sum of coherent states is deemed nonclassical. This classicality-nonclassicality divide leads to the following natural definition, inspired by the notion of entanglement breaking channels:

Definition: A channel Γ is said to be *nonclassicality-breaking* if and only if the output state $\hat{\rho}_{\rm out} = \Gamma(\hat{\rho}_{\rm in})$ is classical *for every* input state $\hat{\rho}_{\rm in}$, i.e., if and only if the diagonal 'weight' function of every output state is a genuine probability distribution.

Now, the close connection between nonclassicality and entanglement alluded to earlier raises a related and important second issue : *what is the connection, if any, between entanglement-breaking channels and nonclassicality-breaking channels*? To appreciate the nontriviality of this second issue, it suffices to simply emphasize that the very definition of entanglement-breaking refers to bipartite states whereas the notion of nonclassicality-breaking makes no such reference. In this paper we show that both these issues can be completely answered in the case of bosonic Gaussian channels. To this end we first derive the *nonclassicality-based* canonical forms for Gaussian channels.

Let $\mathcal{S}$ denote an element of the symplectic group $Sp(2n, R)$ of linear canonical transformation and $\mathcal{U}(\mathcal{S})$ the corresponding unitary (metaplectic) operator. When

Canonical form	Nonclassicality-breaking condition	Entanglement-breaking condition	Complete-positivity condition
$(\kappa\,\mathbb{1},\,\mathrm{diag}(a,b))$	$(a-1)(b-1)\geq\kappa^4$	$ab\geq(1+\kappa^2)^2$	$ab\geq(1-\kappa^2)^2$
$(\kappa\,\sigma_3,\,\mathrm{diag}(a,b))$	$(a-1)(b-1)\geq\kappa^4$	$ab\geq(1+\kappa^2)^2$	$ab\geq(1+\kappa^2)^2$
$(\mathrm{diag}(1,0),\,Y)$,	$a,\,b\geq1$, $a,\,b$ being eigenvalues of Y	$ab\geq1$	$ab\geq1$
$(\mathrm{diag}(0,0),\,\mathrm{diag}(a,b))$	$a,\,b\geq1$	$ab\geq1$	$ab\geq1$

Table 1: Here $\mathbb{1}$ is the 2×2 identity matrix, and σ_3 is the diagonal Pauli matrix.

one is looking for aspects that are invariant under local unitary operations such as entanglement, it is clear that a Gaussian channel Γ is 'equivalent' to $\mathcal{U}(\mathcal{S}^{'})\,\Gamma\,\mathcal{U}(\mathcal{S})$, for arbitrary symplectic group elements $\mathcal{S},\,\mathcal{S}^{'}\in Sp(2n,\,R)$. The orbits or double cosets of equivalent channels in this sense are the ones classified and enumerated by Holevo and others [2]. The canonical forms so determined are useful, for instance, in the study of entanglement-breaking Gaussian channels.

The classification of Holevo and collaborators is *entanglement-based*, and so it is not suitable for our purpose, since the notion of nonclassicality-breaking has a more restricted local invariance. A nonclassicality-breaking Gaussian channel Γ preceded by any Gaussian unitary $\mathcal{U}(\mathcal{S})$ is nonclassicality-breaking if and only if Γ itself is nonclassicality breaking. In contradistinction, nonclassicality-breaking aspect of Γ and that of $\mathcal{U}(\mathcal{S})\,\Gamma$ [Γ followed the Gaussian unitary $\mathcal{U}(\mathcal{S})$] are not equivalent in general. They are equivalent if and only if $\mathcal{S}$ is in the intersection $Sp(2n,\,R)\cap SO(2n,\,R)$ of symplectic phase space rotations, or passive elements in the quantum optical sense. In the single-mode case this intersection is just the rotation group $SO(2)\subset Sp(2,\,R)$. We thus need to classify single-mode Gaussian channels Γ into orbits or double cosets $\mathcal{U}(\mathcal{R})\,\Gamma\,\mathcal{U}(\mathcal{S})$, $\mathcal{S}\in Sp(2,\,R)$, $\mathcal{R}\in SO(2)\subset Sp(2,R)$. Equivalently, we classify (X,Y) into orbits $(\mathcal{S}\,X\,\mathcal{R},\,\mathcal{R}^T\,Y\,\mathcal{R})$. It turns out that there are three distinct canonical forms for $(X,\,Y)$, shown in Table 1.

With these nonclassicality-based canonical forms of (X,Y) on hand, we now derive the necessary and sufficient conditions for a single-mode Gaussian channel to be nonclassicality-breaking. For channels with nonsingular X, we first arrive at a sufficient condition on the channel parameters to ensure nonclassicality-breaking by asking as to when will the channel transform the input state's 'diagonal weight' function to a valid 'Q' function, the Q function being always pointwise nonnegative in the complex plane. We then derive a necessary condition by looking at the signature of the output diagonal weight function for a particular input state evaluated at a particular phase space point at the output. Surprisingly, we find these two conditions to be one and the same. The proof for channels with singular X is obtained in an analogous manner. The canonical forms and the corresponding necessary and sufficient conditions for nonclassicality-breaking are listed in Table 1. The conditions for entanglement-breaking and complete-positivity are also listed for comparison.

As evident from Table 1, it is clear that the nonclassicality-breaking condition is stronger than the entanglement-breaking condition for all the three canonical forms. Thus, a nonclassicality-breaking channel is necessarily entanglement-breaking. But there are channel parameter ranges wherein the channel is entanglement-breaking, though not nonclassicality-breaking. But this turns out to be a 'weak' failure: if at all the output of an entanglement-breaking channel is nonclassical, the nonclassicality is of a 'weak' kind in the following sense. For every entanglement-breaking channel, there exists a particular value of squeeze-parameter r_0, depending only on the channel parameters and not on the input state, so that the entanglement-breaking channel followed by unitary squeezing of extent r_0 always results in a nonclassicality-breaking channel. It is in this precise sense that nonclassicality-breaking channels and entanglement-breaking channels are essentially one and the same.

To conclude, we have explored the notion of nonclassicality-breaking and its relation to entanglement-breaking [3]. We have shown that the two notions are effectively equivalent in the context of bosonic Gaussian channels, even though at the level of definition the two notions are quite different, the latter requiring reference to a bipartite system. Our analysis shows that some nonclassicality could survive an entanglement-breaking channel, but this residual nonclassicality would be of a particular weaker kind: Not only the output nonclassicality is no more than squeezing-type nonclassicality, but also, and perhaps more importantly, the nonclassicality of all output states can be removed by *one fixed* squeezing transformation. Though we have presented details of the analysis only in the case of single-mode bosonic Gaussian channels, we believe the analysis is likely to generalize to the case of n-mode channels in a reasonably straight forward manner.

References

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