Strong converse for the classical capacity of entanglement-breaking and Hadamard channels

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Abstract. A strong converse theorem for the classical capacity of a quantum channel states that the probability of correctly decoding a classical message converges to zero in the limit of many channel uses, if the rate of communication exceeds the classical capacity of the channel. Along with a corresponding achievability statement for rates below the capacity, such a strong converse theorem enhances our understanding of the capacity as a very sharp dividing line between possible and impossible rates of communication. Here, we show that such a strong converse theorem holds for the classical capacity of all entanglement-breaking channels and Hadamard channels. Prior results regarding strong converse theorems for particular covariant channels emerge as a special case of this approach.

Keywords: classical capacity, strong converse, relative entropy

Introduction.—One of the most fundamental tasks in quantum information theory is the transmission of classical data over many independent uses of a quantum channel, such that, for a fixed rate of communication, the error probability of the transmission decreases to zero in the limit of many channel uses. The maximum rate at which this is possible for a given channel is known as the classical capacity of the channel. Holevo, Schumacher, and Westmoreland (HSW) characterized the classical capacity of a quantum channel $\mathcal{N}$ in terms of the following formula:

$$\chi\left(\mathcal{N}\right) \equiv \max_{\{p_X(x),\rho_x\}} I\left(X;B\right)_\rho, \tag{1}$$

where $\{p_X\left(x\right),\rho_x\}$ is an ensemble of quantum states, $I\left(X;B\right)_\rho \equiv H\left(X\right)_\rho + H\left(B\right)_\rho - H\left(XB\right)_\rho$ is the quantum mutual information, and $H\left(\sigma\right) \equiv -\mathrm{Tr}\{\sigma\log\sigma\}$ is the von Neumann entropy. In the above formula, the quantum mutual information $I\left(X;B\right)$ is computed with respect to the following classical-quantum state:

$$\rho_{XB} \equiv \sum_x p_X\left(x\right)|x\rangle\langle x|_X \otimes \mathcal{N}_{A\to B}\left(\rho_x\right), \tag{2}$$

for some orthonormal basis $\{|x\rangle\}$, and the notation $\mathcal{N}_{A\to B}$ indicates that the channel accepts an input on the system A and outputs to the system B.

For certain quantum channels, the HSW formula is equal to the classical capacity of the channel. These results follow because the Holevo formula was shown to be additive for these channels, in the sense that the following relation holds for these channels for any positive integer n:

$$\chi\left(\mathcal{N}^{\otimes n}\right) = n\,\chi\left(\mathcal{N}\right). \tag{3}$$

However, in general, if one cannot show that the HSW formula is additive for a given channel, then our best characterization of the classical capacity is given by a regularized formula:

$$C(\mathcal{N}) = \chi_{\mathrm{reg}}\left(\mathcal{N}\right) \equiv \lim_{n\to\infty}\frac{1}{n}\chi\left(\mathcal{N}^{\otimes n}\right). \tag{4}$$

The work of Hastings suggests that the regularized limit is necessary unless we are able to find some better characterization of the classical capacity, other than the above one given by HSW. Also, an important implication of the Hastings result, which demonstrates a strong separation between the classical and quantum theories of information, is that using entangled quantum codewords between multiple channel uses can enhance the classical capacity of certain quantum channels, whereas it is known that classically correlated codewords do not.

Given the above results, one worthwhile direction is to refine our understanding of the classical capacity of channels for which the HSW formula is additive. Indeed, the achievability part of the HSW coding theorem states that as long as the rate of communication is below the classical capacity of the channel, then there exists a coding scheme such that the error probability of the scheme decreases exponentially fast to zero. The

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converse part of the capacity theorem makes use of the well known Holevo bound, and it states that if the rate of communication exceeds the capacity, then the error probability of any coding scheme is bounded away from zero in the limit of many channel uses.

Such a converse statement as given above might suggest that there is room for a trade-off between error probability and communication rate. That is, such a "weak" converse suggests that it might be possible for one to increase communication rates by allowing for an increased error probability. A *strong converse theorem* leaves no such room for a trade-off—it states that if the rate of communication exceeds the capacity, then the error probability of any coding scheme converges to one in the limit of many channel uses. Importantly, a strong converse theorem establishes the capacity of a channel as a very sharp dividing line between which communication rates are possible or impossible in the limit of many channel uses.

Strong converse theorems hold for all discrete memoryless classical channels, and the error probability is known to converge to 0 exponentially, thanks to the work of Wolfowitz and later Arimoto. Much later, Polyanskiy and Verdú generalized the Arimoto approach in a very useful way, by showing how to obtain a bound on the success probability in terms of any relative-entropy-like quantity satisfying several natural properties.

Less is known about strong converses for quantum channels: Winter and Ogawa and Nagaoka independently developed a strong converse theorem for channels with classical inputs and quantum outputs. For such channels, the HSW formula in (1) is equal to the classical capacity.

After this initial work, Koenig and Wehner proved that the strong converse holds for the classical capacity of particular covariant quantum channels Their proof is in the spirit of Arimoto—they considered a Holevo-like quantity derived from the Rényi relative entropy and then showed that this quantity is additive for particular covariant channels. This reduction of the strong converse question to the additivity of an information quantity is similar to the approach of Arimoto, but the situation becomes more interesting for the case of quantum channels since entanglement between channel uses might lead to the quantity being non-additive.

Summary of results.—We first prove that a strong converse theorem holds for the classical capacity of all entanglement-breaking channels. Such channels can be modeled as the following process:

1. The channel performs a quantum measurement on the incoming state.
2. The channel then prepares a particular quantum state at the output depending on the result of the measurement.

The channels are said to be entanglement-breaking because if one applies a channel in this class to a share of an entangled state, then the resulting bipartite state is a separable state, having no entanglement. As important subclasses of the entanglement-breaking channels occur the classical-quantum channels mentioned above and quantum measurement channels, in which only the first step above occurs and the output is classical.

Our second result is the strong converse for Hadamard channels, which can be defined as the complementary channels of entanglement-breaking channels, with respect to the Stinespring dilation.

The main result can be stated as follows: for a general entanglement-breaking or Hadamard channel $\mathcal{N}$, the following bound on the success probability for any coding scheme of rate $R > \chi(\mathcal{N})$ and any block length n holds:

$$p_{\text{succ}} \leq 2^{-nc(R)}, \tag{5}$$

where $c(R) > 0$ depends on the rate R and the channel.

Full details of definitions and proofs can be found in Ref. [1].

Conclusion.—We have proven a strong converse theorem for the classical capacity of all entanglement-breaking and Hadamard channels, building on tighter bounds on the success probability in terms of a "sandwiched" Rényi relative entropy (cf. [2]). Our approach also allows us to recover the earlier results of Koenig and Wehner. This information measure should find other applications in quantum information theory, given that many other information measures can be obtained from a relative entropy.

References

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