When do Local Operations and Classical Communication Suffice for Optimal Quantum State Discrimination?

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Abstract. In this talk we consider when an ensemble of states can be optimally distinguished by local operations and classical communication (LOCC). It is shown that almost all two-qubit ensembles consisting of three pure states cannot be optimally discriminated using LOCC. This is surprising since *any* two pure states can be distinguished optimally by LOCC. Additionally, we prove an easy sufficient condition for when a set of product states cannot be optimally distinguished by LOCC. We also provide an example of N-party LOCC for which optimal identification is not possible for any finite N, however optimality is achievable locally as $N \to \infty$.

Keywords: quantum state discrimination, LOCC, separable operations, non-locality without entanglement

1 Introduction

A fundamental topic in quantum information is the problem of *state discrimination*, which investigates how well ensembles of quantum states can be distinguished under various physical conditions. One important operational setting in which questions of distinguishability emerge is the so-called "distant lab" scenario. Here, some multiparty quantum state is distributed to spatially separated quantum labs, and the various parties use local measurements combined with classical communication to try and identify their state. This operational setting is also known as LOCC (Local Operations and Classical Communication), and the study of LOCC operations has played an important role in developing our understanding of not only quantum information processing [1], but also the nature of quantum entanglement itself [2].

As LOCC operations are just a subset of all possible physical operations, certain state discrimination tasks become impossible when the distant lab constraint is imposed. In general it is a very challenging problem to decide whether or not a particular set of states can be optimally distinguished using LOCC. This is due to the complex structure of a general LOCC operation in which, due to the global communication, the choice of local measurement by one party at one particular round can depend on the measurement outcomes of all the other parties in previous rounds. It is thus helpful to visualize a general LOCC operation as a tree where each node indicates a particular choice of local measurement and each branch corresponds to a particular sequence of measurement outcomes. Deciding whether or not a certain discrimination task is feasible by LOCC therefore amounts to a consideration of all such possible trees.

Despite its complexity, partial progress has been made in understanding the capabilities and limitations of LOCC state discrimination. Most notably is the discovery that *any* two orthogonal pure states can be perfectly distinguished using LOCC [3]. A similar result holds for pairs of non-orthogonal states in which again, LOCC can obtain the optimal discrimination success probability that is physically possible [4]. This finding is particularly relevant to the current talk since we will show that, in sharp contrast, almost all triples of two qubit states *cannot* be optimally distinguished by LOCC.

The fact that non-LOCC measurements can distinguish certain ensembles better than any LOCC strategy may not be overly surprising when the ensemble states possess entanglement. This is because entanglement embodies some non-local property of two or more system, and thus some global measurement across all systems is needed in general to discriminate among entangled states. However, rather surprisingly, certain ensembles exist that consist of unentangled states that cannot be distinguished optimally using LOCC [5]. This phenomenon is often called "nonlocality without entanglement," and it essentially reflects that fact that nonlocality and entanglement are two different physical properties of multipartite quantum systems. Understanding the difference between the two is an important problem in quantum information science, and thus a main objective of this talk is to study, in particular, LOCC discrimination of product state ensembles.

2 Summary of our results

In this talk, we begin by returning to the problem of *perfect* state discrimination among two-qubit orthogonal states. While our primary interest is LOCC discrimination, we will also consider discrimination by the more general class of separable operations (SEP). The two-qubit perfect discrimination problem has been solved for almost all types of ensembles. Our first contribution is that we solve the missing piece of perfect discrimination between one pure state and one mixed state by either LOCC and SEP. Interestingly, we find that SEP is more powerful than LOCC in the sense certain ensembles are

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distinguishable by SEP but not LOCC. For instance, we obtain the following:

Corollary 1 *For $2 \otimes 2$ systems, let $|\psi\rangle$ be orthogonal to some mixed state ρ, with a maximally entangled state $|\Phi\rangle$ being orthogonal to both. Then $|\psi\rangle$ and ρ are perfectly distinguishable by SEP. On the other hand, they are perfectly distinguishable by LOCC iff $|\psi\rangle$ is either a product state or maximally entangled.*

We next move on to investigate the problem of *minimum-error* discrimination between linearly independent states. However, we prove that this seemingly more general problem actually reduces to the problem of perfect discrimination of orthogonal states. This reduction therefore allows us to apply the results of the latter toward the problem of minimum-error discrimination among non-orthogonal (linearly independent) states. For three orthogonal two-qubit pure states, Walgate and Hardy have proven that LOCC perfect discrimination is possible iff at least two of the states are product states [6]. As the set of product states occupy zero volume in state space, we are able to prove our main result.

Theorem 2 *Three randomly chosen two-qubit pure states almost surely cannot be optimally discriminated by LOCC.*

When focusing on product state ensembles, we are able to obtain a simple necessary condition for when three product states cannot be distinguished optimally by LOCC.

Theorem 3 *Suppose that $\{|\psi_\lambda\rangle := |\alpha_\lambda\rangle|\beta_\lambda\rangle, p_\lambda\}_{\lambda=1}^3$ ($p_\lambda > 0$) is some linearly independent two-qubit product state ensemble that spans $\{|\Phi\rangle\}^\perp$. Let $\lambda_{min}(\Phi)$ denote the smallest squared Schmidt coefficient of $|\Phi\rangle$. If*

$$p_i^2 \lambda_{min}^2(\Phi) > p_j^2 |\langle\psi_i|\psi_j\rangle|^2 + p_k^2 |\langle\psi_i|\psi_k\rangle|^2$$

for every choice of i, j, k such that $\{i, j, k\} = \{1, 2, 3\}$, then the ensemble cannot be distinguished optimally (in the minimum error sense) with LOCC.

With this result, new examples of nonlocality without entanglement can easily be constructed.

We also move beyond two qubit ensembles and consider the optimal discrimination of three symmetric N-qubit states. The specific ensemble we analyze is the N-copy generalization of the celebrated double trine ensemble [7, 8]. Specifically, this is the equiprobable ensemble of N-qubit states $\{|\psi_i\rangle\}_{i=0}^2$ with $|\psi_i\rangle = (U^i|0\rangle)^{\otimes N}$, where $U = -e^{2\pi i/N} \begin{pmatrix} 1/2 & -\sqrt{3/4} \\ \sqrt{3/4} & 1/2 \end{pmatrix}$. We prove that for any finite N, the ensemble cannot be optimally discriminated using N-party LOCC. However as $N \to \infty$, we present a protocol that indeed achieves optimal (perfect) discrimination. This is quite different from the N-copy discrimination among two possible states which can always be accomplished optimally by LOCC [9].

Finally, we consider the task of unambiguous discrimination by LOCC. We obtain new upper bounds on the success probability obtainable by any LOCC measurement for ensembles of two-qubit symmetric states.

Theorem 4 *Let $\{|\psi_i\rangle, p_i\}_{i=1\ldots3}$ be an ensemble of two-qubit linearly independent symmetric pure states with $|\widetilde{\psi}_i\rangle$ being dual states satisfying $\langle\widetilde{\psi}_i|\psi_j\rangle = 0$ for $i \neq j$. If $C(\widetilde{\psi}_i) \geq |\langle\widetilde{\psi}_i|\psi_i\rangle|^2$ for all i, then LOCC cannot obtain an unambiguous probability greater than* $\max\{p_1, p_2, p_3\}$.

With this simple examples can be found when LOCC is insufficient for optimal unambiguous discrimination.

On the other hand, if we consider the equiprobable ensemble of the double trine states, then the conditions of Theorem 4 are not met. In fact, there exists an LOCC POVM that obtains a success probability of 1/2. This is the greatest LOCC success probability since we are able to prove that it is also the greatest success probability obtainable by SEP; however both of these are less than the optimal probability feasible by global operations. This finding is interesting since for the task of minimum-error discrimination separable operations are strictly stronger than LOCC [8]. Thus, the double trine ensemble demonstrates a very curious distinguishability property: For minimum-error discrimination, LOCC $<$ SEP $=$ GLOBAL; For optimal unambiguous discrimination, LOCC $=$ SEP $<$ GLOBAL. This raises the intriguing question of whether there exists certain ensembles for which LOCC$<$GLOBAL with respect to one performance measure but LOCC$=$GLOBAL with respect to another.

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