TOPOLOGY I

ASSIGNMENT DUE ON 13 AUGUST 2012

(1) Show that, for any prime number p, $\mathbf{Z}$ with the metric $d(x, y) = |x - y|_p$ is a metric space. Recall that $|a|_p = p^{-v}$, where v is the greatest non-negative integer for which p^v divides a.

(2) In the metric space of (1), show that $B(x, \epsilon)$ is infinite for all $x \in \mathbf{Z}$ and $\epsilon > 0$.

(3) The *lower limit topology* on $\mathbf{R}$ (denoted $\mathbf{R}_l$) is the topology where half-open intervals of the form $[a, b)$ form a basis for the topology.

   (a) Show that open sets in the usual topology of $\mathbf{R}_l$ are open in the lower limit topology.

   (b) Show that a function $f : \mathbf{R}_l \to \mathbf{R}$ is continuous if and only if, for each $a \in \mathbf{R}$, $\lim_{x \to a+} f(x) = f(a)$.

   (c) Is there a metric on $\mathbf{R}$ which gives rise to this topology?

(4) Show that the L^p metrics, $1 \leq p \leq \infty$ on $\mathbf{R}^n$ all give rise to the same topology.

(5) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\cap_\alpha \mathcal{T}_\alpha$ is a topology on X. What about $\cup_\alpha \mathcal{T}_\alpha$?

(6) Let $\{T_\alpha\}$ be a family of topologies on X. Show that there exists a unique smallest topology containing all the the topologies $\mathcal{T}_\alpha$ and a unique largest topology contained in all the topologies $\mathcal{T}_\alpha$.

(7) Show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbf{R}$.

(8) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b,\ a \text{ and } b \text{ rational}\}$$

generates a topology *different* from the lower limit topology on $\mathbf{R}$.

(9) Show that the countable collection

$$\{(a, b) \times (c, d) \mid \ a < b \text{ and } c < d,\ a, b, c, d \text{ rational}\}$$

is a basis for the topology of $\mathbf{R}^2$.