$\mathbb{Q}\left[\binom{n}{2}\right]$

k_0, k_1, k_2

$$k_i(x,y) = \begin{cases} 1 & \text{if } |x \cap y| = i \\ 0 & \text{otherwise.} \end{cases}$$

$$T_i f(x) = \sum_{y \in \binom{n}{2}} k_i(x,y)\, f(y).$$

$$T_i T_j = T_{k_i * k_j}$$

$$k_i * k_j(x,y) = \sum_{z} k_i(x,z)\, k_j(z,y)$$

	$\cancel{0}$ 2	1	$\cancel{0}$ 2
2 $\cancel{0}$	1	0	0
	0	1	0
	0	0	1
1	0	$2(n-2)$	0
	1	$n-2$	$n-3$
	0	4	$2(n-4)$
$\cancel{2}$	0	0	$\binom{n-2}{2}$
	0	$n-3$	$\binom{n-3}{2}$
	1	$2(n-4)$	$\binom{n-4}{2}$.

Commutativity of $\mathrm{End}_{S_n} \mathbb{C}[\binom{n}{2}]$:

$$k_i * k_j (x, y)$$

$$= \#\{ z \mid |z \cap x| = i, |z \cap y| = j \} =: S_1$$

$$\overset{?}{=} \#\{ z \mid |z \cap x| = j, |z \cap y| = i \} =: S_2$$

<u>H.W</u>: Construct a bijection from S_1 to S_2.

$$V = V_1^{\oplus m_1} \oplus V_2^{\oplus m_2} \oplus \cdots \oplus V_k^{\oplus m_k}$$

$$\mathrm{End}_{S_n} V = \bigoplus_{i=1}^{k} M_{m_i \times m_i}(\mathbb{Q}) \quad \boxed{\mathrm{End}_{S_n} V_i = \mathbb{Q}}$$

So $\mathrm{End}_{S_n} V$ is commutative iff $m_i \leq 1$ for each i.

$$\mathrm{End}\left(\mathbb{C}[\tbinom{n}{2}]\right) = \bigoplus_{i=1}^{\leq 3} \mathbb{C}$$

$$\overset{??\ ??}{\underbrace{V_0}_{\text{const. fns.}} \oplus V_1 \oplus V_2}$$

<u>Recall</u>: ~~$\mathbb{C}[\binom{n}{2}]$~~ $\mathbb{C}[\underline{n}] = \underbrace{V_0}_{\text{simple const fns.}} \oplus \underbrace{\mathbb{C}[n]_0}_{\text{simple}}$

$$k_1 * k_1(\{1,2\},\{1,2\})$$

$$= \#\{s \mid |s \cap \{1,2\}| = 1\}$$

$$= 2\binom{n-2}{1} = 2(n-2)$$

$$k_1 * k_1(\{1,2\},\{1,3\})$$

$$= \#\{s \mid |s \cap \{1,2\}| = |s \cap \{1,3\}| = 1\}$$

$$= (n-3) + 1 = n-2.$$

$$k_1 * k_1(\{1,2\},\{3,4\})$$

$$= \#\{s \mid |s \cap \{1,2\}| = |s \cap \{3,4\}| = 1\}$$

$$= 4$$

$$k_1 * k_0(\{1,2\},\{1,2\})$$

$$= \#\{s \mid s \cap \{1,2\} = 1, \} = 0.$$

$$k_1 * k_0(\{1,2\},\{1,3\})$$

$$= (n-3)$$

$$k_1 * k_0(\{1,2\},\{3,4\})$$

$$= 2(n-4)$$

$k_0 * k_0(\{12\}, \{12\})$

$= \#\{S \mid S \cap \{1,2\} = 0\} = \binom{n-2}{2}$

$k_0 * k_0(\{1,2\}, \{1,3\})$

$= \#\{S \mid S \cap \{1,2\}, S \cap \{1,3\}\} = \binom{n-3}{2}$

Matrix of right mult. by T_2:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by T_1:

$$\begin{pmatrix} 0 & 2(n-2) & \\ 1 & n & \\ 0 & & \end{pmatrix} \qquad \begin{pmatrix} 0 & 2(n-2) & 0 \\ 1 & n-2 & n-3 \\ 0 & 4 & 2(n-4) \end{pmatrix}$$

by T_0:

$$\begin{pmatrix} 0 & 0 & \binom{n-2}{2} \\ 0 & n-3 & \binom{n-3}{2} \\ 1 & 2(n-4) & \binom{n-4}{2} \end{pmatrix}$$

$$T_1^2 = 16T_2 + 8T_1 + 4T_0. \qquad (n=10).$$

$$T_1^2 = 2(n-2)T_2 + (n-2)T_1 + 4T_0$$

$$T_1^3 = \cancel{16T_2T_1}$$

$$2(n-2)\cancel{T_2}T_1 + (n-2)T_1^2 + 4T_0T_1$$

$$= 2(n-2)T_1 + (n-2)\left[2(n-2)T_2 + (n-2)T_1 + 4T_0\right]$$

$$+ 4\left[(n-3)T_1 + 2(n-4)T_0\right]$$

$$= 2(n-2)^2\cancel{T_2} + \left[(n-2)^2 + 2(n-2) + 4(n-3)\right]T_1$$

$$+ \left[4(n-2) + 8(n-4)\right]T_0\left[\cancel{T_1^2 - 8T_1 - 16}\right]$$

$$T_1^2 - 2(n-2)T$$

$$\mathrm{End}_{S_n}\left(\mathbb{Q}\left(\left[{n \atop 2}\right]\right)\right) \cong \mathbb{Q}[t] \big/ (t+2)(t-6)(t-16)$$

$$\cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}.$$

$$-\beta(t+2) + \alpha(t-6)(t-16) = 1$$

$$p(t) = \alpha(t-6)(t-16) = 1 + \beta(t+2)$$

$$t+2 \,\big)\, t^2 - 22t + 96 \,\big(\, t - 24$$

$$t^2 + 2t$$

$$-24t - 96$$

$$-24t - 48$$

$$-48$$

$$\frac{(t+2)(t-24)}{48} \cancel{-248} = \cancel{\frac{(t-6)(t-16)}{48}}$$

$$= 1$$

$$- \frac{1}{48}(t-6)(t-16)$$

$$\alpha = -\frac{1}{48}, \quad -\beta = \frac{t-24}{48}$$

$$p_{-2}(t) = -\frac{(t-6)(t-16)}{48}$$