V, W K - alg. closed

Schur's lemma:

$$\dim_K \mathrm{Hom}_G(V,W) = \begin{cases} 1 & \text{if } W \cong V \\ 0 & \text{otherwise.} \end{cases}$$

$$\dim_K \mathrm{Hom}_G(V_1 \oplus V_2, W_1 \oplus W_2) = \begin{cases} 4 & \text{if } V_1 \cong V_2 \cong W_1 \cong W_2 \\ 2 & \\ 1 & \\ 0 & \end{cases}$$

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$A : V_1 \to W_1$ $\quad B : V_2 \to W_1$

$C : V_1 \to W_2$ $\quad D : V_2 \to W_2$

$$T(v_1 \oplus v_2) = \left(A(v_1) + B(v_2)\right) \oplus \left(C(v_1) + D(v_2)\right)$$

$$T_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} ; \quad T_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

$$T_1 \circ T_2 = \begin{pmatrix} A_1 A_2 + B_1 C_2 & A_1 B_2 + B_1 D_2 \\ C_1 A_2 + D_1 C_2 & C_1 B_2 + D_1 D_2 \end{pmatrix}$$

matrix multiplication.

T is an intertwiner $\Longleftrightarrow$ A, B, C, D are intertwiners.

$$V = V_1^{\oplus m_1} \oplus V_2^{\oplus m_2} \oplus \cdots \oplus V_k^{\oplus m_k}$$

$$W = V_1^{\oplus n_1} \oplus V_2^{\oplus n_2} \oplus \cdots \oplus V_k^{\oplus n_k}$$

~~two~~ $V_1, V_2, \ldots, V_k$: pairwise non-isomorphic simple.

$$\mathrm{Hom}_G(V,W) = \bigoplus_{i=1}^{k} \mathrm{Hom}_G(V_i^{\oplus m_i}, V_i^{\oplus n_i})$$

$$= \bigoplus_{i=1}^{k} M_{n_i \times m_i}(K)$$

$$\dim \mathrm{Hom}_G(V,W) = \sum m_i n_i$$

~~$m_1 = 2,\ n_1 = 1,\ m_2 = 1,\ n$~~

$$\mathrm{Lin}(K^m, K^n) \longleftrightarrow M_{n\times m}(K)$$

$$T(v_1 \oplus \cdots \oplus v_{m_i}) = \sum a_{11}(v_1) + a_{12}(v_2) + \cdots$$

$$\oplus$$

$$\sum a_{21}(v_1) + a_{22}(v_2) + \cdots$$

$$\oplus$$

$$\vdots$$

$G \circlearrowright X$

$K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

$\rho(g) : K[X] \to K[X]$

$\rho(g) f(x) = f(g^{-1}x)$

$K[X]$

$$\langle f, g \rangle = \sum_{x \in X} f(x) g(x) \quad (K = \mathbb{Q}, \mathbb{R})$$

$$\sum_{x \in X} f(x) \overline{g(x)} \quad (K = \mathbb{C})$$

$$\langle \rho(g) f_1, \rho(g) f_2 \rangle = \langle f_1, f_2 \rangle$$

$$\| $$

$$\sum f_1(g^{-1}x) \overline{f_2(g^{-1}x)} = \sum f_1(x) f_2(x)$$

$\rho(g)$ preserves this inner product.

<u>Nice consequences:</u>

① $\rho(g) V \subseteq V \Rightarrow \rho(g) V^\perp \subseteq V^\perp$

$V^\perp = \{ w \in V \mid \langle w, v \rangle = 0 \ \forall v \in V \}$

② Orthogonal complement of invariant subspace is again invariant.

Every permutation rep. over $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{C}$ is a sum of simple representations.

$C_2 \circlearrowright \{0,1\} = X$

$\mathbb{F}_2[X]$

$K[X]$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{matrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} ; +1 \\ \\ \begin{pmatrix} 1 \\ -1 \end{pmatrix} ; -1 \end{matrix}$$

$$K^2 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$$

Suppose $\begin{pmatrix} a \\ b \end{pmatrix}$ spans an invariant subspace

$\Rightarrow$ $\begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector with eigenvalue 1.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$a+b=0$

$\Rightarrow a=b$.

<u>Maschke's thm:</u> $|G|$ is not div. by char(K) then every finite dim rep. of G over K is a sum of simples.

$\mathrm{End}_G V_i = K$ (V_i ~~is~~ splits over K)

$$V = V_1^{\oplus m_1} \oplus \cdots \oplus V_k^{\oplus m_k}$$

$$T \in \mathrm{End}_G V = \bigoplus_{i=1}^{k} M_{m_i \times m_i}(K)$$

$$T = \alpha_1 \oplus \cdots \oplus \alpha_k$$

where $\alpha_i \in M_{m_i \times m_i}(K)$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ \varepsilon_i = \mathrm{Id}_{K^{m_i}} \end{array}$$

$$\begin{array}{l} 1 \in \mathrm{End}_G V \\ \parallel \\ \varepsilon_1 \oplus \cdots \oplus \varepsilon_k \leftarrow \text{very special} \end{array}$$

$$(\alpha_1 \oplus \cdots \oplus \alpha_k)(\beta_1 \oplus \cdots \oplus \beta_k)$$

$$= \alpha_1\beta_1 \oplus \cdots \oplus \alpha_k\beta_k$$

$$\varepsilon_i(V) = V_i^{\oplus m_i}$$

<u>R : ring</u>

<u>Defn</u>: Primitive central idempotents.

- central : z is central if $zr = rz$ $\forall r \in R$
- idempotent : $\varepsilon^2 = \varepsilon$

A central idempotent ε is said to be primitive if $\varepsilon = \varepsilon_1 + \varepsilon_2$ where ε_1 & ε_2 are ~~orthogonal~~ central idempotents then $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$

$T_k f(x) = \sum k(x,y) f(y)$

$$(T_{k_1} \circ T_{k_2}) f(x)$$

$$= \sum_y k_1(x,y)\,(T_{k_2} f)(y)$$

$$= \sum_{y,z} k_1(x,y)\, k_2(y,z)\, f(z)$$

Let $k(x,z) = \sum k_1(x,y) k_2(y,z)$
$\quad \overset{\shortparallel}{k_1 * k_2(x,z)}$

$$T_{k_1} \circ T_{k_2} = T_k .$$

$\underline{\text{Ind } End_{S_n} K[\underline{n}]}$:

$$k_1(i,j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o/w}. \end{cases}$$

$$k_2(i,j) = \begin{cases} 0 & \text{if } i=j \\ 1 & \text{o/w} \end{cases}$$

$$k_1 * k_1(i,j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} = k_1$$

$$k_1 * k_1(i,j) = \sum_j k_1(i,l)\, k_1(l,j) = 1$$

$$k_1 * k_2$$

$T_k f(x) = \sum k(x,y) f(y)$

$$(T_{k_1} \circ T_{k_2}) f(x)$$

$$= \sum_y k_1(x,y) (T_{k_2} f)(y)$$

$$= \sum_{y,z} k_1(x,y) k_2(y,z) f(z)$$

Let $k(x,z) = \sum k_1(x,y) k_2(y,z)$

$\quad\; \shortparallel$

$\; k_1 * k_2(x,z)$

$$T_{k_1} \circ T_{k_2} = T_k .$$

Ind $\mathrm{End}_{S_n} K[\underline{n}]$:

- $k_1(i,j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o/w} \end{cases}$.

$k_2(i,j) = \begin{cases} 0 & \text{if } i=j \\ 1 & \text{o/w} \end{cases}$

$k_1 * k_1(i,j) = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} = k_1$

$k_1 * k_1(i,j) = \sum_j k_1(i,j) k_1(j,j) = 1$

$k_1 * k_2$

$\cancel{\lambda_1 + \dots + \lambda_m = n}$

$k \leq n$

$S_n \circlearrowright X_k \; \binom{n}{k} = \{ S \subseteq \underline{n} : |S| = k \}$

Get permutation representations:

$S_n \circlearrowright K\left[\binom{n}{k}\right]$

$$S_n \circlearrowright K\left[\binom{n}{1}\right] = K[\underline{n}] = K \oplus \underset{\substack{\uparrow \\ \text{zero-sum fns.} \\ \text{simple.}}}{K[\underline{n}]_0}$$

$$\mathrm{End}_{S_n}(K[\underline{n}]) := \mathrm{Hom}_{S_n}(K[\underline{n}], K[\underline{n}])$$

$$= \{ T_k \mid k : \underline{n} \times \underline{n} \to K ,\ k(\omega i, \omega j) = k(i,j) \}$$

So $\dim \mathrm{End}_{S_n}(K[n]) = ?$

$$= \dim \{ k : \underline{n} \times \underline{n} \to K \mid k(\omega i, \omega j) = k(i,j) \ \forall \omega \in S_n, i,j \in \underline{n} \}.$$

$$= |S_n \backslash \underline{n} \times \underline{n}| = 2$$

$G \circlearrowright X$

$$\dim \{ k : X \to K \mid k(gx) = k(x) \ \forall g \in G, x \in X \}$$

$$= |G \backslash X|$$

$V = \mathbb{k}\left[\binom{n}{k}\right]$

$$\mathrm{End}_{S_n} V = \left\{ T_k \;\middle|\; k : S_n \backslash \binom{n}{k} \times \binom{n}{k} \longrightarrow \mathbb{K} \right\}$$

Given $\underline{i} = \{i_1, \dots, i_k\}$, $\underline{i}' = \{i'_1, \dots, i'_k\}$
$\underline{j} = \{j_1, \dots, j_k\}$; $\underline{j}' = \{j'_1, \dots, j'_k\}$

~~Does~~ when does there exist $w \in S_n$

such that
$$w \cdot \underline{i} = \underline{i}'$$
$$w \cdot \underline{j} = \underline{j}'$$

All you need is : $|\underline{i} \cap \underline{j}| = |\underline{i}' \cap \underline{j}'|$

If $k \leq \frac{n}{2}$.

then $|\underline{i} \cap \underline{j}| \in \{0, \dots, k\}$.

Conclusion: If $k \leq \frac{n}{2}$, then

$$\dim_{S_n} \mathbb{k}\left[\binom{n}{k}\right] = k .$$

$$k_1 * k_2 (i,i) = \sum_j k_1(i,j)\, k_2(j,i)$$

$$= \cancel{\sum} \; k_1(i,i)\, k_2(i,i) = 0$$

$$k_1 * k_2 (i,j) = \sum_{l} k_1(i,l)\, k_2(l,j)$$

$$= k_2(i,j) = 1$$

$$k_1 * k_2 = k_2$$

$$k_2 * k_1 = k_2$$

$$k_2 * k_2 (i,i) = \sum_{j} k(i,j)\, k(j,i)$$

$$= (n-1)$$

$$k_2 * k_2 (i,j) = \sum_{l} k(i,l)\, k(l,j)$$

$$= n-2 .$$

$$k_1 * k_1 = k_1$$

$$k_1 * k_2 = k_2 = k_2 * k_1$$

$$k_2 * k_2 = (n-1) k_1 + (n-2) k_2 .$$

$$k_1 = \varepsilon_1 + \varepsilon_2$$