(ρ, V) $\quad V$: f.d. v.s. over an algebrically closed field. finite dim.

<u>Lemma</u> (Schur) : If V is simple, and $T: V \to V$ is an intertwiner (i.e., $\rho(g) \circ T = T \circ \rho(g) \;\; \forall g \in G$) then $T = \lambda I$ for some scalar λ.

<u>Example:</u> Two dim rep of S_4 (L10)

$$\rho(s_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(s_1) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

<u>Fact:</u> If A commutes with B, then A preserves the eigenspaces of B.

$v \in V_\lambda$, i.e., $Bv = \lambda v$

$BAv = ABv = \lambda Av$ so $Av \in V_\lambda$

So T commutes with $\rho(s_2) \Rightarrow$

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

commutes with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \alpha_1 = \alpha_2$$

so $T = \alpha \, \mathrm{id}$.

$T : V \to V$

Since the field is alg. closed, T has an eigenvalue λ, and so a

$$V_\lambda := \{v \in V \mid Tv = \lambda v\} \neq \{0\}.$$

$$V_\lambda = \ker(T - \lambda I)$$

Since $T - \lambda I$ is an intertwiner, V_λ is an invariant subspace.

By simplicity of V, $V_\lambda = V$

$\Rightarrow \quad T = \lambda I \qquad \mathrm{End}_a V = \underset{\substack{\uparrow \\ \text{field}}}{K} \cdot \mathrm{Id}_V$

<u>Schur's lemma II</u> Suppose V_1 & V_2 are both simple, and $T : V_1 \to V_2$ is a non-zero intertwiner. Then every intertwiner $S : V_1 \to V_2$ is of the form $S = \lambda T$ for some scalar λ

<u>Pf:</u> $T^{-1} \circ S = \lambda \, \mathrm{Id}_{V_1}$.

In general (if the field is not algebraically closed, then $\mathrm{End}_k V$ can be a division algebra).

$\rho : C_n \to \cancel{\mathbb{R}^2}\ \mathbb{Q}^2$

$$\rho(1) = \begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix}$$

Qn: What is $\mathrm{End}_{C_n} \mathbb{Q}^2$?

Qn: Is the converse of Schur's lemma true? ~~Df~~ (If $\mathrm{End}_G V = \lambda \mathrm{Id}_V$ then V is simple)

Example: If G ~~an~~ finite abelian group, k - alg. closed then $\{\rho(g) \mid g \in G\} \subset \mathrm{End}\, V$ ~~are all diagonalizable.~~ are also pairwise commut~~ative~~ing and diagonalizable.

$A^k = I$ for some k,

then ~~&~~ A satisfies $f(x) = 0$

where $f(x) = x^k - 1 = (x-\lambda_1)\dots(x-\lambda_k)$

which has distinct roots.

$\Rightarrow$ A is diagonalizable

$$G = C_4 \overset{\text{"}\{0,1,2,3\}\text{"}}{} \text{ - gen by } 1$$

<u>Rep</u>: $\rho(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{R}^2$

Who commutes with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} -b & a \\ -d & c \end{pmatrix} = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$$

$$c = -b \quad a = d$$

$$T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$\mathrm{End}_G V = \{T : V \to V \mid T \text{ is intertwiner}\}$$

is a ring.

In our example:

$$\mathrm{End}_{C_4} \mathbb{R}^2 = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} \cong \mathbb{C}.$$

$$a + ib : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (ax - by) + i(ay + bx)$$
$$x + iy \mapsto (a+ib)(x+iy)$$

matrix : $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

If all these matrices $\rho(g), g \in G$ are scalar matrices, then

V simple $\Rightarrow$ $\dim V = 1$

If not, $\exists\, g_0 \in G \ni \rho(g_0)$ is not scalar.

$\therefore$ $\exists$ eigenvalue λ such that $0 \neq V_\lambda \neq V$

V_λ will be preserved by all $\rho(g), g \in G$

(because $\rho(g)$ commute with $\rho(g_0)$)

So V simple $\Rightarrow \dim V = 1$.

<u>Have:</u> Every <u>simple</u> rep. over <u>alg. closed</u> field of an ~~finite~~ <u>ab. gp</u> is one dim.

~~<u>Necessity of finiteness</u>:~~

$$\cancel{G = \mathbb{Z}}$$

$$\cancel{\rho(1) = (}$$

<u>Free group on two generators</u>

$F_2 = G = \langle x, y \rangle$

element - a word in letters x, y, x^{-1}, y^{-1}
including the empty word.

with x not next to x^{-1}, y not next to y^{-1}

~~product~~ group law = concatenation

Reps of F_2 $\longleftrightarrow$ pairs of invertible linear maps $S, T : V \to V$

$$S = \rho(x)$$
$$T = \rho(y)$$

$V = \mathbb{C}^{\mathbb{N}}$ = space of sequences of cx. nos.

$$\{x_1, x_2, x_3, \dots\}$$

$$S : \{x_n\} \mapsto \{0, x_1, x_2, \dots\}$$
$$T : \{x_n\} \mapsto \{x_2, x_3, x_4, \dots\}.$$

$$TS = \mathrm{Id}$$
$$ST = \{0, x_2, x_3 \dots\}$$

W is an invar. subspace for S & T, it also invar. for $TS - ST = \{x_1, 0, \dots\}$

$$\rho(x) = I + S$$
$$\rho(y) = I + T$$