$$A_n = \{ w \in S_n \mid inv(w) \text{ is even} \}$$

$$\left.\begin{aligned} \varepsilon &: S_n \longrightarrow \{\pm 1\} \\ \varepsilon &: w \longmapsto (-1)^{inv(w)} \end{aligned}\right\} \text{ homom.}$$

$$A_n = \ker(\varepsilon)$$

$w_0$ : any odd permutation

$$w \mapsto w w_0$$

$$A_n \underset{\cong}{\overset{\cong}{\rightleftarrows}} S_n \setminus A_n$$

$$w \circ x = w x w^{-1}$$

So $|A_n| = \dfrac{n!}{2}$.

---

Suppose $S_n \circlearrowright X$, so $A_n \circlearrowright X$

Every $S_n$ orbit is a union of $A_n$-orbits.

$$\begin{aligned} A_n \circ x \subset\ & S_n \circ x \\ & \parallel \\ & (A_n \amalg A_n w_0) \cdot x \\ & \parallel \\ & A_n \circ x \cup A_n (w_0 x) \end{aligned}$$

Two cases:

1. $w_0 x \in A_n \cdot x$

In this case: $A_n \cdot x = S_n \cdot x$

2. $w_0 x \notin A_n \cdot x$

In this case: $\left.\begin{array}{l} A_n \cdot x, \\ A_n \cdot (w_0 x) \end{array}\right\}$ two $A_n$ orbits

make up $S_n \cdot x$.

Example of case 1

$x = 1 \quad w_0 = (12)$

$w_0 x = 2 = (123) \cdot x \in A_n x$

---

Take $X = \binom{n}{2}$.

What are $S_n$-orbits? $w_0 = (12$

$x = \{1, 2\} \qquad w_0 x = x$

$A_n \cdot x = S_n \cdot x = \binom{n}{2}$

In general this works for $\binom{n}{k}$

$S_n \circlearrowright S_n$

$w \cdot x = w x w^{-1}$

Orbits $\leftrightarrow$ partitions of $n$.

$\underline{\underline{S_3}}$ $\quad \lambda = (3) \quad (1\,2\,3)$

$\lambda = (2,1) \quad (1\,2)$

$\lambda = (1,1,1) \quad e\ id.$

$w_0 = (1\,2)$

$w_0 (1\,2\,3) w_0^{-1} \stackrel{?}{=} a (1\,2\,3) a^{-1}$

for some $a \in A_3$.

$w_0 a (1\,2\,3)(w_0 a)^{-1} = (1\,2\,3)$

Does there exist an odd permutation that commutes with $(1\,2\,3)$?

$(w_0 a(1), w_0 a(2), w_0 a(3)) = (1\,2\,3)$

$w_0 a$ can be $\left.\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}\right\}$ all even

so $S_n \cdot (1\,2\,3)$ is a union of 2 $A_n$-orbits.

$x \in S_n$, $S_n \circlearrowright S_n$ by conjugation

$S_n \cdot x = A_n \cdot x$

$wxw^{-1} = x \Leftrightarrow wx = xw$

$\Leftrightarrow$ $x$ commutes with an odd permutation.

---

$n=4$

~~(4)~~ ✓

(3,1) ✗

(2,2) ✓

~~(2,1,1)~~ ✓

(1,1,1,1) ✓

$A_4$ has 4 classes

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(5) ✗

~~(4,1)~~

~~(3,2)~~

(3,1,1) ✓

(2,2,1) ✓

(2,1,1,1) ✓

(1,…,1) ✓

(1 2 3)(4)(5)

(1 2 3 4)

$A_5$ has 5 classes.

For partitions of 6 only (5,1) & (3,3) are suspect.

guilty

$(1,2,3)(4,5,6)$

$w_0 = (1,4)(2,5)(3,6)$

## Conjugacy classes in $A_n$

$\Longleftrightarrow$ ~~par~~ 2 classes ~~to~~ for each partn with distinct odd parts.

1 class for all other even partition.

$\lambda = (\lambda_1, \ldots, \lambda_m)$ even

$\Leftrightarrow$ even no. of $\lambda_i$ are even.