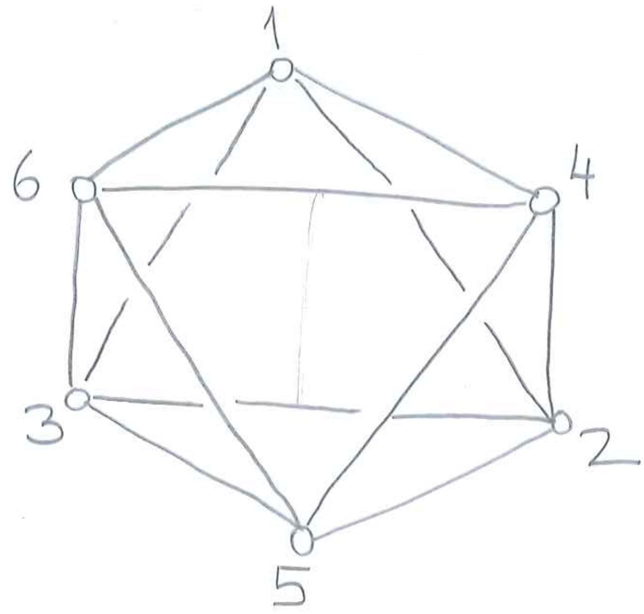
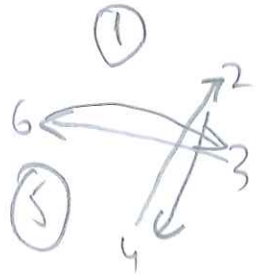
HAPPY OCTAHEDRON DAY! (1



Case 1



$f_1 = (123, 456)$

$f_2 = (146, 235)$

$f_3 = (124, 356)$

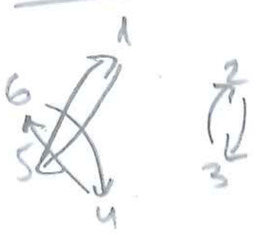
$f_4 = (136, 245)$

$s_1 = (15)(23)(46)$

$s_2 = (14)(26)(35)$

$s_3 = (15)(24)(36)$

Case 2

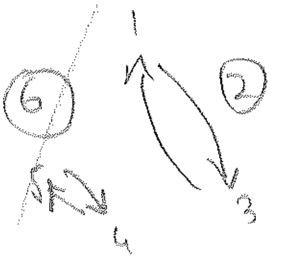


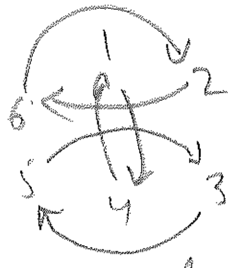
Case 1		Case 2	
123 → 146		123 → 235	
124 → 124		124 → 356	
136 → 136		136 → 245	
456 → 235		456 → 146	
356 → 356		356 → 124	
245 → 245		245 → 136	

For S_2,

$146 \longrightarrow 356$
$235 \longrightarrow 124$
$123 \longrightarrow 123$
$136 \longrightarrow 136$

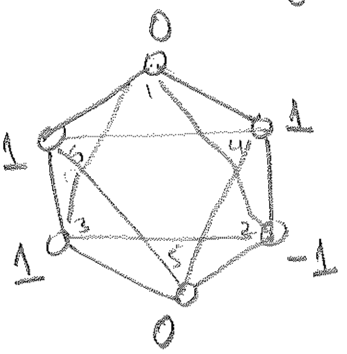
$146 \longrightarrow 324$
$235 \longrightarrow 356$
$123 \longrightarrow 456$
$136 \longrightarrow 245$

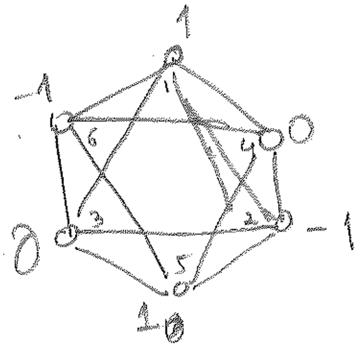


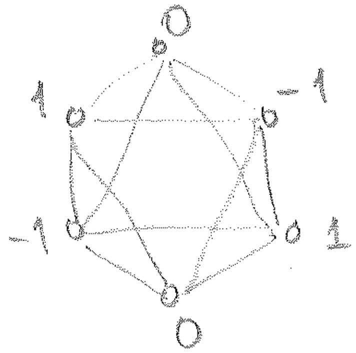


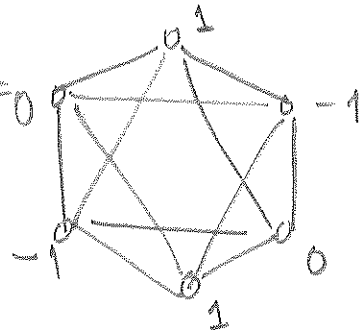
Embedding of $S_4 \hookrightarrow S_6$ as ~~icosahedral~~ octahedral group:

$$(12) \longmapsto (15)(23)(46)$$
$$(23) \longmapsto (14)(26)(35)$$
$$(34) \longmapsto (15)(24)(36).$$

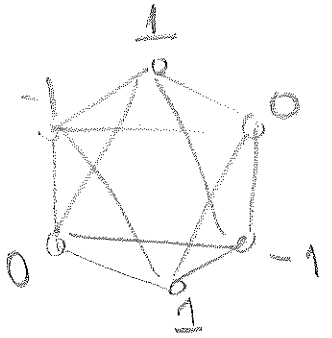
$v_1 =$ 

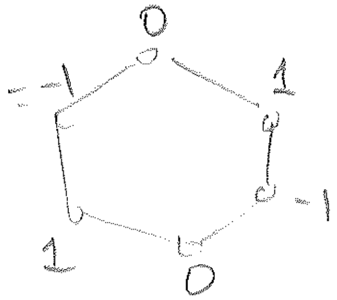
$v_2 =$ 

$s_1 v_1 =$  $= -v_1$

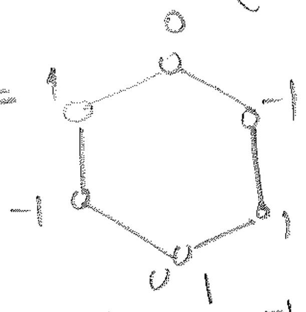
$s_1 v_2 =$  $= v_2 - v_1$

$$\rho(s_1) = \begin{pmatrix} -1 & -1 \\ 0 & +1 \end{pmatrix}$$

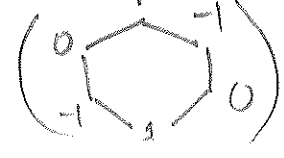
$s_2 v_1 =$  $= v_2$

$s_2 v_2 =$  $= v_1$

$$\rho(s_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$s_3 v_1 =$  $= -v_1$

$$\rho(s_3) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

$s_3 v_2 =$  $= v_2 - v_1$

Recall: S_3 has an irred. 2 dim rep. W.

Spanned $(1,-1,0) = w_1$, $(0,1,-1) = w_2$, S_3 acts by permutation of coordinates.

$$s_1 w_1 = -w_1$$

$$s_1 w_2 = (1,0,-1) = (1,-1,0) + (0,1,-1) = w_1 + w_2$$

$$\sigma(s_1) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}. \qquad s_2 w_1 = (1,0,-1)$$

$$\sigma(s_2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Goal: to find an iso. $V \xrightarrow{T} W$

$$T\rho(s_i) = \sigma(s_i) T \qquad i = 1, 2.$$

$$T\rho(s_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\parallel$$

$$\sigma(s_2)\, T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix}$

$$T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \frac{\alpha+\beta}{2} \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \frac{\alpha-\beta}{2} \end{pmatrix}$$

$$T = \begin{pmatrix} \alpha & \alpha \\ \frac{\alpha+\beta}{2} & \frac{\alpha-\beta}{2} \end{pmatrix} \quad \begin{pmatrix} \alpha & \alpha \\ \alpha-\varepsilon & \alpha+\varepsilon \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 1-\varepsilon & 1+\varepsilon \end{pmatrix}$$

Assume $\alpha = 1$

$$\begin{pmatrix} 2 & 2 \\ 1-\varepsilon & 1+\varepsilon \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 1-\varepsilon & 1+\varepsilon \end{pmatrix}$$

$\cancel{1-\varepsilon}$ $\quad -1+\varepsilon+1+\varepsilon$

$2\varepsilon = 1+\varepsilon \qquad \varepsilon = 1$

$$T = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{??}{=} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \checkmark$$

$$\cancel{\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \overset{??}{=} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

$$V\downarrow_{S_3} = W$$

$$V \cong V \otimes \varepsilon$$

$$W \cong W \otimes \varepsilon$$