$G \curvearrowright X$ & $G \curvearrowright Y$ finite sets

$k : X \times Y \longrightarrow \mathbb{Q}$

$T_k : \mathbb{Q}[Y] \longrightarrow \mathbb{Q}[X]$

$$T_k f(x) = \sum_{y \in Y} k(x,y) f(y)$$

$$\left(T_k \rho_Y(g) f\right)(x) \overset{?}{=} \left(\rho_X(g) T_k f\right)(x)$$

$$\left(T_k \rho_Y(g) f\right)(x) = \sum_{y \in Y} k(x,y)\, \rho_Y(g) f(y) = \sum_{y \in Y} k(x,y)\, f(g^{-1}y) = \sum_{y \in Y} k(x, gy)\, f(y)$$

$$\left(\rho_X(g) T_k f\right)(x) = T_k f(g^{-1}x) = \sum_{y \in Y} k(g^{-1}x, y)\, f(y)$$

$$k(x, gy) = k(g^{-1}x, y) \quad \forall\, x \in X, y \in Y, g \in G$$

$$k(gx, gy) = k(x,y) \quad \forall\, x \in X, y \in Y, g \in G$$

$S_n \circlearrowleft \underline{n} \rightsquigarrow \mathbb{Q}[\underline{n}]$. n dim rep. of S_n

$$\mathbb{Q}[\underline{n}] = \underset{\text{const fns.}}{\mathbb{Q}} \oplus \underset{\text{zero-sum fns.}}{V}$$

$S_n \circlearrowleft \binom{n}{2} \rightsquigarrow \mathbb{Q}\left[\binom{n}{2}\right]$. $\binom{n}{2}$-dim rep of S_n.

$$\mathbb{Q}\left[\binom{n}{2}\right] = \underset{\text{const fns.}}{\mathbb{Q}} \oplus \underset{\text{zero-sum}}{V}$$

Can we break it up further?

<u>Intertwiner:</u> (ρ, V) (σ, W)

$$T : V \longrightarrow W$$

$$T\rho(g)v = \sigma(g)Tv \quad \forall g \in G, \forall v \in V.$$

<u>Lemma:</u> If T is an intertwiner, then $\mathrm{Im}(T)$ is an invariant subspace of W.

<u>Pf</u> $\sigma(g)T(v) = T\rho(g)v$

<u>Example:</u> $X = \binom{n}{2}$ $\quad Y = \underline{n}$.

$$k : \binom{n}{2} \times \underline{n} \longrightarrow \mathbb{Q}$$

$$k(\omega \cdot S, \omega \cdot i) = k(S, i) \quad \forall\, S \in \binom{n}{2},\ \forall\, i \in \underline{n}.$$

$$k(S, i) = 1 \quad \forall\, S \in \binom{n}{2},\ i \in \underline{n}$$

$\underline{n} = 4$

$$S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$$

$$\underline{4} = \{1,2,3,4\}$$

$$T_k f(S) = \sum_{i \in \underline{n}} f(i) \qquad \text{const. fns.}$$

$$k(S,i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

$$T_k f(S) = \sum_{i \in S} f(i)$$

$\underline{\underline{n=4}}$ $\mathbb{C}[4]$ has basis $\delta_1, \delta_2, \delta_3, \delta_4$

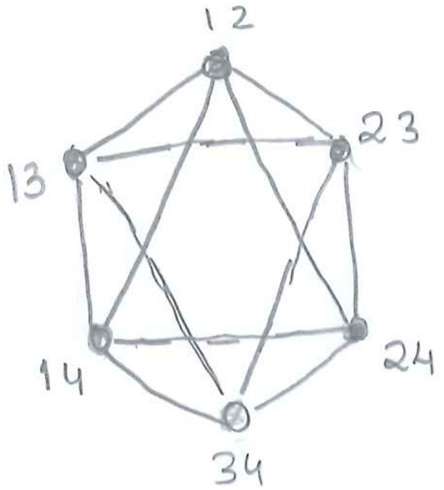
$$T_k : \delta_i \mapsto \cancel{\sum_{S\ni i} f(S)} \quad \sum_{S\ni i} \delta_S$$

$$T_k \delta_i(S) = \sum_{j\in S} \delta_i(j) = \begin{cases} 1 & \text{if } i\in S \\ 0 & \text{if } i \notin S. \end{cases}$$

$\underline{\text{Qn}}$: What is the dimension of $T_k(\mathbb{Q}[\underline{n}])$?

~~12, 1,4~~
12, 14, 24 $\to$ 3
13, 23, 34 $\to$

$\binom{n}{2} \xrightarrow[\text{stru}]{\text{graph}}$



octahedral graph.

Octahedral graph!!

Face $= \{(S_1, S_2, S_3) \mid S_i \cap S_j \neq \emptyset \ \forall\, i,j\}$

~~X = pairs of disjoint faces of $\binom{n}{2}$~~
complementary

$$\rho \otimes \varepsilon(g) = \rho(g) \cdot \varepsilon(g)$$

$$\varepsilon(g) = (-1)^{\mathrm{inv}(g)}$$

$$\varepsilon(g_1 g_2) = \varepsilon(g_1)\varepsilon(g_2)$$

<u>Ex:</u> Show that $V_2 \cong V_2 \otimes \varepsilon$,

i.e., <u>find a</u> linear iso $T: V_2 \to V_2$

such that $\cancel{\rho(\omega)T + T\rho(\omega) = 0}$

$$\rho(\omega)T = \varepsilon(\omega)\, T\, \rho(\omega)$$

for all $\omega \in S_4$.

$$Tf(S) = f(S^c)$$

$T: V \to V$ is identity fn.!

$$\cancel{\rho(s_1)(x,y) = (x, x+y)}$$

$X =$ faces of $\binom{n}{2}$ whose intersection is trivial.

$$\mathbb{Q}\left[\binom{n}{2}\right] \supseteq \text{span}\{\text{indicator char. fun of non-complementary faces}\}$$

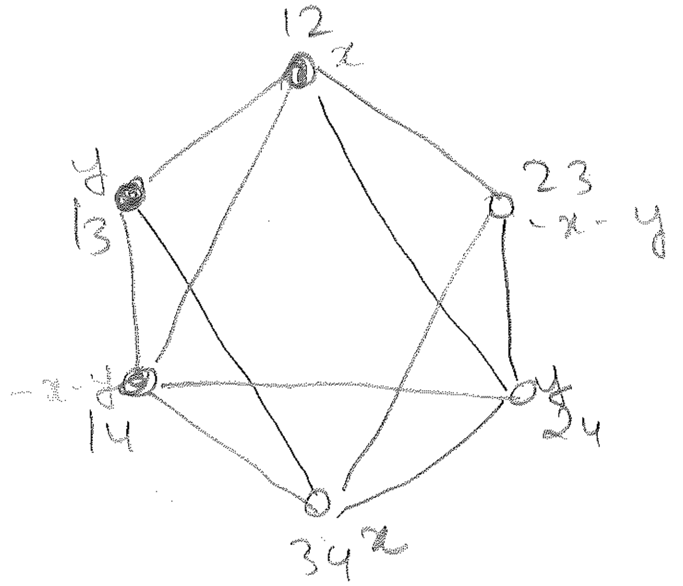
$$\text{by } \|.$$

$T_k\delta_1 = \delta_{12} + \delta_{13} + \delta_{14}$ span{char fun of complementary faces}

$T_k\delta_2 = \delta_{12} + \delta_{23} + \delta_{24}$.

$T_k\delta_3 = \delta_{13} + \delta_{23} + \delta_{34}$

$T_k\delta_4 = \delta_{14} + \delta_{24} + \delta_{34}$.



$(n,4)$

$$V_2 = \{ f : \text{Octahedral graph} \to \mathbb{Q} \mid f(\triangledown) = 0 \}$$

this is a two dimensional rep. of S_4

$$\mathbb{C}\binom{n}{2} = \mathbb{Q} \oplus V_1 \oplus V_2$$