MIDSEMESTER EXAMINATION SOLUTIONS

LOCALLY COMPACT ABELIAN GROUPS

(1) Let L be a locally compact abelian group which admits a compact open subgroup. Let G be a closed subgroup of L which is isomorphic to T^m for some cardinal number m. There exists a closed subgroup G' of L such that the map $G \times G' \to L$ defined by $(x, x') \mapsto x + x'$ is an isomorphism of topological groups.

Proof. Let K be a compact open subgroup of L. Then G is contained in K (or else $G/G \cap K$ would be a disconnected image of G, which is connected). Since Cartesian powers of T split from compact abelian groups, G splits from K. Let $s : K \to G$ be a continuous splitting section (i.e., s is a continuous homomorphism and $s(g) = g$ for all $g \in G$). Since G is divisible, s extends to a homomorphism $\tilde{s} : L \to G$, which is also continuous, since it is continuous on K. Let $G' = \ker \tilde{s}$. Clearly, $G \cap G' = 0$. Also, any $x \in L$ can be written as $x = \tilde{s}(x) + (x - \tilde{s}(x))$. It follows that the continuous map $\phi : G \times G' \to L$ given by $(x, x') \mapsto x + x'$ is a bijection. By the open mapping theorem, $\phi_{|G \times G' \cap K}$ is open, and it obviously maps onto K. In other words, $\phi^{-1} : L \to G \times G'$ is continuous when restricted to K. Therefore ϕ^{-1}, being a homomorphism that is continuous at 0, is continuous everywhere. □

(2) There is only one compact topology on $\mathbf{Z}_p$.

Proof. Let $\mathbf{Z}_p$ be endowed with a compact topology. For each $n \in \mathbf{N}$, $K_n := p^n \mathbf{Z}_p$ is the image of $\mathbf{Z}_p$ under multiplication by p^n, and is therefore a compact subgroup. Since it is of finite index, it is also open. It follows that the topology is stronger than the topology of $\mathbf{Z}_p$ where the K_n form a local base at 0. The Pontryagin dual is therefore a subgroup of the Prüfer group. Since the Prüfer group has no infinite proper subgroups, the dual must be the entire Prüfer group. It follows from Pontryagin duality that $\mathbf{Z}_p$ is isomorphic to the dual of the Prüfer group.

In the above proof, the inference that the K_n's are open can be obtained from the open mapping theorem if we assume that $\mathbf{Z}_p$ is a countable union of compact sets, giving the slightly stronger result that there is only one topology on $\mathbf{Z}_p$ with respect to which it is a countable union of compact sets. □

(3) Let p and p' be distinct primes. Let A be a discrete p-torsion group. Let G be a compact topological p'-group.

(a) Multiplication by p is an automorphism of G.

(b) $\mathcal{E}(A, G) = 0$.

Proof of (a). Multiplication by p has an inverse on $\hat{G}$ (which is a p'-torsion group), and hence an inverse on G. $\square$

Proof of (b). Let L be an extension over A with kernel G (we may forget about topology). Define $s : L \to G$ as follows: given $x \in L$ choose n such that $p^n x \in G$. Let $s(x) = p^{-n}(p^n x)$ (here p^{-n} denotes the n-fold application of the inverse of the multiplication by p map of G). One easily verifies that s is a splitting section. $\square$

(4) Let $A = \prod_p \mathbf{Z}/p\mathbf{Z}$, a product over all primes.
   (a) A/A_{tor} is divisible.
   (b) $0 \to A_{\text{tor}} \to A \to A/A_{\text{tor}} \to 0$ does not split.

Proof of (a). A_{tor} consists of elements of A whose coordinates are 0 at all but finitely many p. Given $x = (x_p) \in A$ and $N \in \mathbf{N}$ let

$$y_p = \begin{cases} x_p/N & \text{if } (p, N) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(x_p/N makes sense in $\mathbf{Z}/p\mathbf{Z}$ if $(p, N) = 1$). Let $y = (y_p)$. Since only finitely many primes divide N, $Ny - x \in A_{\text{tor}}$. $\square$

Proof of (b). If $x = (x_p) \in A$ and $x \neq 0$, then $x_p \neq 0$ for some prime p, so that $x \neq py$ for any $y \in A$. Thus A has no non-trivial infinitely divisible elements, and therefore no subgroup isomorphic to A/A_{tor}. $\square$